DEPARTMENT OF ECONOMICS

Working Paper

Asymptotic Bias of OLS in the Presence of Reverse Causality

by

Deepankar Basu

Working Paper 2015-18

UNIVERSITY OF MASSACHUSETTS
AMHERST

Asymptotic Bias of OLS in the Presence of Reverse Causality

Deepankar Basu*

November 21, 2015

Abstract

In this paper, I derive an expression for the asymptotic bias in the OLS estimator of the partial effect of a regressor on the dependent variable when there is reverse causality and all variables in the model are covariance stationary. I show that the sign of the asymptotic bias depends only on the signs of the bi-directional causal effects.
JEL Codes: C10, C30.
Keywords: reverse causality; simultaneity bias.

1 Introduction

In applied econometric work, it is common to encounter models where there is bi-directional causality. This is especially common in macroeconomic contexts where key variables are jointly determined, but also in microeconometric contexts, where there are two-way causal mechanisms at work. In such cases, not only does the key regressor have a causal effect on the dependent variable, but a casual effect runs in the other direction – from the dependent variable to the same regressor – too.

*Department of Economics, University of Massachusetts, 1012 Thompson Hall, Amherst, MA 01003, email: `dbasu@econs.umass.edu`. I would like to thank Michael Ash for comments on an earlier version of this note.

Some common examples of reverse causality are: (a) the relationship between public debt (regressor) and economic growth (dependent variable) across countries (Panizza and Presbitero, 2014); (b) the relationship between per capita real income (regressor) and life expectancy at birth (dependent variable) across countries or states (Pritchett and Summers, 1996); (c) the relationship between public expenditure on health care (regressor) and health outcomes (dependent variable) across countries (Filmer and Pritchett, 1999). Other examples could be estimation of demand-supply systems, estimation of a Keynesian consumption function (where endogeneity emerges from the equilibrium condition), estimation of the effect of crime control on the incidence of crime, and many other such cases.

While it is obvious that the OLS estimator of the partial effect of the regressor on the dependent variable is biased and inconsistent due of endogeneity, the direction or magnitude of the bias is not immediately clear. But in such cases, knowing the direction of the bias can be useful. For instance, if the sign of the bias is positive (negative) then we can assert that the OLS estimator provides an upper (lower) bound for the true partial effect. This short note derives an expression for the asymptotic bias for a general model of reverse causality, and shows that it is possible to know its sign when we have information about the signs of the bi-directional causal effects.

The rest of the paper is organized as follows: in the next section I set out the model and derive the main results. The proof of the main proposition is given in an appendix.

2 Model and Results

To fix ideas, let us posit a bi-directional causal relationship between two covariance stationary processes y_{it} and z_{it} in a panel data setting. We will capture this two-way causal relationship in terms of two structural relationships. The first captures the causal effect of z_{it} on y_{it},

$$y_{it} = z_{it}\alpha_1 + \mathbf{u'}_{it}\boldsymbol{\beta_1} + \epsilon_{1,it}, \tag{1}$$

and the second captures the causal relationship running in the opposite direction from y_{it} to z_{it},

$$z_{it} = y_{it}\alpha_2 + \mathbf{v'}_{it}\boldsymbol{\beta_2} + \epsilon_{2,it}. \tag{2}$$

In (1) and (2), $i = 1, 2, \ldots n$ indexes units (e.g., cities, states, countries), $t = 1, 2, \ldots T$ indexes years, $\mathbf{u'}_{it} = (1, u_{1,it}, u_{2,it}, \ldots, u_{k-1,it},)$ and $\mathbf{v'}_{it} = (1, v_{1,it}, v_{2,it}, \ldots, v_{k-1,it},)$ are k-vectors of covariance stationary, strictly exogenous variables (including a constant), $\boldsymbol{\beta_1}, \boldsymbol{\beta_2}$ are vectors of parameters, and $\epsilon_{1,it}, \epsilon_{2,it}$ are structural errors with

$$\begin{pmatrix} \epsilon_{1,it} \\ \epsilon_{2,it} \end{pmatrix} \sim \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$

To ensure the existence of the reduced form models, we will need

Assumption 1. *For the models in (1) and (2), we have* $\alpha_1\alpha_2 \neq 1$.

To proceed, let us re-write the structural relationships given by (1) and (2) as

$$\begin{bmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{bmatrix} \begin{bmatrix} y_{it} \\ z_{it} \end{bmatrix} = \begin{bmatrix} \mathbf{u'}_{it}\boldsymbol{\beta_1} + \epsilon_{1,it} \\ \mathbf{v'}_{it}\boldsymbol{\beta_2} + \epsilon_{2,it} \end{bmatrix}.$$

As long as the matrix on the left is non-singular, which is guaranteed by assumption 1, we can get the following reduced form system

$$\begin{bmatrix} y_{it} \\ z_{it} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{u}'_{it}\boldsymbol{\beta_1} + \epsilon_{1,it} \\ \mathbf{v}'_{it}\boldsymbol{\beta_2} + \epsilon_{2,it} \end{bmatrix}.$$

This shows that

$$z_{it} = \left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\mathbf{u}'_{it}\beta_{\mathbf{1}} + \left(\frac{1}{1-\alpha_1\alpha_2}\right)\mathbf{v}'_{it}\beta_{\mathbf{2}} + \left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\epsilon_{1,it} + \left(\frac{1}{1-\alpha_1\alpha_2}\right)\epsilon_{2,it},$$

so that

$$E(z_{it}\epsilon_{1,it}) = \left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\sigma_1^2 \neq 0, \tag{3}$$

because $\mathbf{u}'_{it}$ and $\mathbf{v}'_{it}$ are vectors of strictly exogenous variables, $\epsilon_{1,it}$ and $\epsilon_{2,it}$ are uncorrelated (because they are zero mean, structural errors), and we use the fact that $E(\epsilon_{1,it}^2) = \sigma_1^2 \neq 0$. This immediately shows that the OLS estimator of α_1 in (1) will be inconsistent.

To derive an expression for the asymptotic bias, let us replace the full set of exogenous variables in (1) with a constant and a linear combination of the non-constant exogenous regressors as

$$\mathbf{u}'_{it}\boldsymbol{\beta} = \beta_0 + \beta_1 x_{1,it} \tag{4}$$

where

$$x_{1,it} = u_{1,it} + \frac{\beta_2}{\beta_1}u_{2,it} + \cdots + \frac{\beta_{k-1}}{\beta_1}u_{k-1,it}$$

which can always be done without loss of generality by choosing the exogenous regressor with a non-zero coefficient as u_1. Using this, we can re-write (1) as

$$y = \mathbf{W}\boldsymbol{\beta} + \boldsymbol{\epsilon_1} \tag{5}$$

where $\mathbf{W} = (\mathbf{z} \quad 1 \quad \mathbf{x_1})$, with $\mathbf{z}$ a $(N \times 1)$ vector (where $N = n \times T$) and $\mathbf{x_1}$ a $(N \times 1)$ vector, and $\boldsymbol{\beta'} = (\alpha_1 \quad \beta_0 \quad \beta_1)$, with α_1 the scalar parameter of interest (the partial effect of z on y). Our main result about the asymptotic bias in the OLS estimator of α_1 is given as

Proposition 1. *Let $\hat{\alpha}_1$ denote the OLS estimator of α_1 in the model in (1). Let $V(z) > 0$ denote the variance of z and r_{z,x_1} denote the correlation coefficient between z and x_1. Then, the asymptotic bias is given by*

$$plim_{N\to\infty}\hat{\alpha}_1 - \alpha_1 = \frac{1}{V(z)\left\{1 - r^2_{z,x_1}\right\}}\left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\sigma_1^2.$$

The proof of Proposition 1 is given in the Appendix. Here, we can use it to derive signs of the asymptotic bias. To proceed, note that since $\sigma_1^2/\left(V(z)\left\{1 - r^2_{z,x_1}\right\}\right) > 0$, the sign of the asymptotic bias is equal to $\text{sgn}(\alpha_2/(1-\alpha_1\alpha_2))$, where sgn(.) is the sign function. Hence, we can derive the sign of the asymptotic bias for the following cases.

1. Case 1, where α_1 and α_2 in (1) and (2) are of opposite signs: In this case, the sign of the asymptotic bias is equal to $\text{sgn}(\alpha_2)$. This is because $1 - \alpha_1\alpha_2 > 0$.
2. Case 2, where α_1 and α_2 in (1) and (2) are of the same sign: There are two sub-cases.
   (a) Case 2a, where $1-\alpha_1\alpha_2 > 0$: In this case, the sign of the asymptotic bias is equal to $\text{sgn}(\alpha_2)$.
   (b) Case 2b, where $1-\alpha_1\alpha_2 < 0$: In this case, the sign of the asymptotic bias is equal to $\text{sgn}(-\alpha_2)$.

Thus, the sign of the asymptotic bias of the OLS emtimator of α_1 in (1) is equal to either $\text{sgn}(\alpha_2)$ or $\text{sgn}(-\alpha_2)$ depending on whether α_1 and α_2 are of the same sign, and if they have opposite signs, whether their product is less than unity.

References

Filmer, D. and Pritchett, L. (1999). The impact of public spending on health: Does money matter? *Social Science and Medicine*, 49:1309–1323.

Panizza, U. and Presbitero, A. F. (2014). Public debt and economic growth. *Journal of Macroeconomics*, 41:21–41.

Pritchett, L. and Summers, L. H. (1996). Wealthier is healthier. *The Journal of Human Resources*, 31(4):841–868.

Appendix

Here, I give the proof of Proposition 1. Denoting the OLS estimator of $\boldsymbol{\beta}$ in (5) as $\hat{\boldsymbol{\beta}}$, we have

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = \left(\frac{\mathbf{W}'\mathbf{W}}{N}\right)^{-1}\left(\frac{\mathbf{W}'\boldsymbol{\epsilon_1}}{N}\right).$$

Using the expression for $\mathbf{W}$, we see that

$$\frac{\mathbf{W}'\mathbf{W}}{N} = \frac{1}{N}\begin{pmatrix} \sum_i z_i^2 & \sum_i z_i & \sum_i z_i x_{1i} \\ \sum_i z_i & \sum_i 1 & \sum_i x_{1i} \\ \sum_i z_i x_{1i} & \sum_i x_{1i} & \sum_i x_{1i}^2 \end{pmatrix}$$

so that, under standard regularity conditions for the convergence of covariance stationary processes,

$$\text{plim}_{N\to\infty}\frac{\mathbf{W}'\mathbf{W}}{N} = \mathbf{A}^{-1}, \text{ where } \mathbf{A} = \begin{pmatrix} E(z^2) & E(z) & E(zx_1) \\ E(z) & 1 & E(x_1) \\ E(zx_1) & E(x_1) & E(x_1^2) \end{pmatrix}.$$

On the other hand, we have

$$\text{plim}_{N\to\infty}\frac{\mathbf{W}'\boldsymbol{\epsilon_1}}{N} = \begin{pmatrix} \text{plim}_{N\to\infty}\frac{\mathbf{z}'\boldsymbol{\epsilon_1}}{N} \\ \text{plim}_{N\to\infty}\frac{\mathbf{1}'\boldsymbol{\epsilon_1}}{N} \\ \text{plim}_{N\to\infty}\frac{\mathbf{x_1}'\boldsymbol{\epsilon_1}}{N} \end{pmatrix} = \begin{pmatrix} \left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\sigma_1^2 \\ 0 \\ 0 \end{pmatrix}$$

Bringing these together, we get

$$\text{plim}_{N\to\infty}\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = \begin{pmatrix} E(z^2) & E(z) & E(zx_1) \\ E(z) & 1 & E(x_1) \\ E(zx_1) & E(x_1) & E(x_1^2) \end{pmatrix}^{-1} \begin{bmatrix} \left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\sigma_1^2 \\ 0 \\ 0 \end{bmatrix}.$$

Hence,

$$\text{plim}_{N\to\infty}\hat{\alpha}_1 - \alpha_1 = \frac{V(x_1)}{det(\mathbf{A})}\left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\sigma_1^2 \tag{6}$$

where $V(x_1)$ and $det(\mathbf{A})$ denote the variance of x_1 and the determinant of $\mathbf{A}$, respectively. The determinant of $\mathbf{A}$ can be written as

$$\begin{aligned} det(\mathbf{A}) =& E(z^2)\left\{E(x_1^2) - [E(x_1)]^2\right\} - E(z)\left\{E(z)E(x_1^2) - E(x_1)E(zx_1)\right\} \\ &+ E(zx_1)\left\{E(z)E(x_1) - E(zx_1)\right\} \end{aligned}$$

which can be simplified to give

$$det(\mathbf{A}) = V(z)V(x_1)\left\{1 - \left[\frac{E(zx_1) - E(z)E(x_1)}{\sqrt{V(z)V(x_1)}}\right]^2\right\}.$$

Plugging this expression in (6), we get

$$\text{plim}_{N\to\infty}\hat{\alpha}_1 - \alpha_1 = \frac{1}{V(z)\left\{1 - r^2_{z,x_1}\right\}}\left(\frac{\alpha_2}{1-\alpha_1\alpha_2}\right)\sigma_1^2,$$

where r_{z,x_1} is the correlation coefficient between z and x_1. This completes the proof.