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Larger groups may alleviate collective action problems

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Abstract This paper shows how larger group size can enhance punishing behavior in social dilemmas and hence support higher levels of cooperation. We focus on describing conflict technology using Lanchester's equations and study the role of "collectivity" of punishment to support cooperation in large groups. The main results suggest that as long as defectors are, even slightly, less "collective" than punishers, Lanchester's law can be applied to show that a smaller proportion of punishers can successfully eliminate defectors as the size of the population increases.

Keywords: Collective action, group size, collective punishment, Lanchester's law

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1 Introduction

Provisions of public goods in a community, revolutionary activities to overthrow corrupt and inefficient governments, and more generally various actions that members of a group take to achieve a common goal have all been examined extensively by social scientists under the name of collective action. Group size has played an important role in explaining collective action. A standard argument put forward by Olson (1965) asserts that a larger group faces more difficulties in achieving a common goal compared to a smaller group, because of an aggravated free-rider problem: "unless the number of individuals in a group is quite small ... rational, self-interested individuals will not act to achieve their common or group interests"(Olson, 1965, p.2).

Since then, studies of the relationship between group size and the provision of collective goods have been conducted by various researchers (Chamberlin, 1974; Marwell and Pamela, 1993; McGuire, 1974; Oliver and Marwell, 1988; Sandler, 1992; Agrawal and Goyal, 2001; Esteban and Ray, 1999). Chamberlin (1974) emphasizes the distinction between goods with perfect non-rivalness and goods with rivalness of consumption among nonexcludable goods. With this distinction he argues that Olson's claim that larger groups would provide fewer public goods only holds for goods with perfect rivalness. In the case of goods with non-rivalness, Chamberlin shows that as group size increases, the amount of total contribution, in absolute terms, would increase. This view − that the Olson thesis holds when the collective good is private but may be reversed when the good is purely public − was initiated by Chamberlin and substantiated by others (Chamberlin, 1974; Marwell and Pamela, 1993; McGuire, 1974; Oliver and Marwell, 1988; Sandler, 1992); it is called "common wisdom" by Esteban and Ray (2001)

Esteban and Ray (2001) examine the group size effect using a model with intergroup conflicts. In particular, they show that under plausible assumptions about costs, the winning probabilities of a larger group is greater than that of a smaller group even if the prize is purely private. However, the context in which they examine collective action − competition between several groups − may be more relevant in some instances, but it is neither a general situation nor the situation that Olson considers.

A variety of empirical or experimental studies have also examined the group size hypothesis (Isaac and Walker, 1988; Bagnoli and McKee, 1991; Isaac et al., 1994; Hann and Koorema, 2002; Carpenter, 2007) and many of them find that "the size of a group is positively related to its level of collective action"(Marwell and Pamela, 1993, p.38). Hann and Koorema (2002) use data from a candy bar honor system in 166 firms in the Netherlands and find evidence that free riding decreases with group size. Carpenter (2007) tests the group size hypothesis when punishment is allowed and finds that "large groups contribute at rates no lower than small groups because punishment does not fall appreciably in large groups" (Carpenter, 2007, p.31). In sum, even though various empirical and experimental studies suggest that large groups may perform better, few theoretical works provide the logic and reasoning of how larger groups can overcome an aggravated free-rider problem.

We examine critically the traditional understanding of the role of group size in collective action, when members of the population punish defectors in a public goods game. Particularly, we focus on describing conflict technology using Lanchester's equations and study the role of "collectivity" of punishment to support cooperation in large groups. Boehm (1982) introduces and emphasizes the functioning of collective punishment or sanctioning in maintaining social norms:

> ... group sanction emerged as the most powerful instrument for regulation of individually assertive behaviors, particularly those which very obviously disrupted cooperation or disturbed social equilibrium needed for group stability. (Boehm, 1982, p.146)

The idea of using Lanchester's equations in explaining collective action is not new; biologists have been applying Lanchester's law to collective action among animals. For example, Franks and Partridge (1993) use Lanchester's square law to explain why predatory army ants rely on large numbers of workers that are smaller than their prey. In the context of human collective action problems, Bingham (1999, 2000) invokes Lanchester's square law to claim that the cost of punishment decreases exponentially as the number of punishers becomes larger. He argues that the remote killing ability of humans and their precursors – the special capacity of the human species to kill at a distance from its target – enables a large number of punishers to attack a single cheater simultaneously, and hence Lanchester's square law applies.

We observe that Bingham's point is valid only in the situation where a large number of punishers face a single cheater. Because the number of punishers is not always large even in a large group, it is not clear whether the same argument can carry over to the collective action problem in large groups. In addition, there is no reason to expect that only punishers can concentrate on attacking, when punishers and defectors engage in conflict. Thus the remote killing competence may be a necessary, but not sufficient, condition for the large group effect.

In this paper we adopt the collective punishment hypothesis and develop a simple model of a public goods game with punishment. We combine the standard evolutionary model of three behavioral types – cooperator, punisher, and defector – with the Lanchester-type conflict between punishers and defectors (Bowles and Choi, 2002; Bowles and Gintis, 2004; Bingham, 1999, 2000; Panchanathan and Boyd, 2004; Sethi and Somanathan, 2006). We introduce a parameter to capture the degree of "collectivity" among punishers who engage the defectors, and study the conditions for the group size effect. We find that group size has a positive effect in supporting higher levels of cooperation. More surprisingly, we show that as long as the defector type is even *slightly* less collective than the punisher type, the large group effect prevails. Thus, we may conclude that the large group effect is quite robust, considering that the defector type, because of its behavioral disposition, would be reluctant to participate in any type of collective action. The organization of the paper is as follows. Section 2 reminds readers of Lanchester's equations and Lanchester's square law. We present the model in section 3 and discuss implications and extensions of the model in section 4.

2 Lanchester's Law and an Illustrating Example

Suppose that x combatants of army A engage y combatants of army B. The time evolutions of x, y are given by Lanchester's equations (Lanchester, 1916, p.20):

$$\frac{dx}{dt} = -\kappa y, \quad \frac{dy}{dt} = -\phi x \tag{1}$$

where κ and ϕ denote the fighting effectiveness of each army. Equation (1) is derived from the assumption that the number of persons knocked out per unit time is directly proportional to the numerical size of the opposing force; during each unit of time, Δt, the opposing force of magnitude y concentrates on the elimination of Δx, so $\Delta x = y\Delta t$. Engel (1954) and Samz (1972) verify the validity of Lanchester's equation in an actual combat situation where U.S. forces captured the island of Iwo Jima during World War II. The solution of differential equations (1) is well-described by a function, $H(x, y)$:

$$H(x, y) = \frac{\phi}{2}x^2 - \frac{\kappa}{2}y^2, \tag{2}$$

in the following sense. When we evaluate $H(x, y)$ at the solutions of (1), its value only depends on the initial values; i.e. $H(x(t), y(t)) = \frac{\phi}{2}(x(t))^2 - \frac{\kappa}{2}(y(t))^2 = \frac{\phi}{2}x_0^2 - \frac{\kappa}{2}y_0^2$ for all $t > 0$. Using this relation we can construct time paths of all solutions starting from various initial values (see figure 1).

We suppose that one army wins a battle if the other army vanishes first; i.e. army A wins the battle at time T, if $\phi(x(T))^2 > \kappa(y(T))^2$, or equivalently $\phi x_0^2 > \kappa y_0^2$ where x_0 and y_0 are initial values for x and y. So, the result of battles depends on the *squares* of the initial numbers of armies, which is called Lanchester's square law. Similarly, if army B divides evenly and, accordingly, army A engages twice with half $\left(\frac{y_0}{2}\right)$ of the original army B, army A wins both battles if $\phi x_0^2 - \kappa\left(\frac{y_0}{2}\right)^2 > \kappa\left(\frac{y_0}{2}\right)^2$, where the left-hand side represents the square of the remaining combatants in army A after the first engagement. In

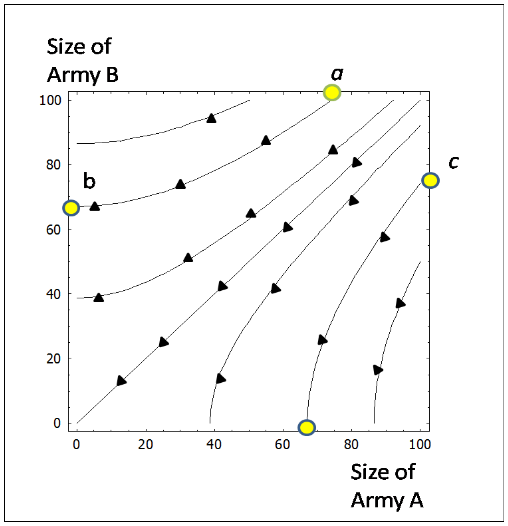

Figure 1: **Solution curves for Lanchester's equation.** Each line corresponds to each solution of Lanchester's equation. If the system starts from a, it will reach b, so army B defeats army A. By constrast, the system, starting from c, predicts the winning of army A. We set $\phi = \kappa = 1$.

general when army B is divided by n we obtain the following rule:

$$\phi x_0^2 > \kappa \frac{y_0^2}{n} \tag{3}$$

Now consider a population which consists of defectors, punishers, and possibly cooperators. For concreteness, suppose that 50% of the population are defectors and suppose that a punishment process – where the punishers eliminate defectors and defectors counteract – is described by Lanchester's equations with $\phi = \kappa$. We assume that the defectors behave individually, so the divisor in the left side of (3), n, equals y_0. Then equation (3) is reduced to $x_0 > \sqrt{y_0}$. First consider the case in which the population size is 20. Since 50% of the population (10 individuals) are defectors, we need 4 punishers to eliminate the defectors ($4 > 3.1328 \approx \sqrt{10}$). However, if the size of the population is 200, only 11 punishers are enough for 100 defectors ($11 > 10 \approx \sqrt{100}$). In other words, when the group size is 20, about 15% of the population must be punishers in order to get rid of the defectors, whereas in a group of size 200, about 5% of the population will be enough.

3 Model and Analyses

3.1 Model

Consider a population of size n playing a public goods game. We suppose that each member in the population can choose to be one of three types: co-operator, punisher, or defector. Punishers (P) contribute to the public project and punish defectors, defectors (D) do not contribute to the public goods, and finally, cooperators (C) do not punish, but only contribute to the public project. A member chooses types taking account of the effect of this choice on the costs he incurs because of ensuing conflicts, which we call a punishment process. In the punishment process, punishers and defectors have a series of engagements described by Lanchester's equations. With the notation $E_i = 1$ if i contributes and $E_i = 0$ otherwise, member i's expected payoffs of each type (or evaluation of utility upon adopting each type) reads

$$\pi(P) = \frac{b}{n} \sum_{j} E_j - c - d \Pr(I_P) \tag{4}$$

$$\pi(D) = \frac{b}{n} \sum_{j \neq i} E_j - s \Pr(I_D) \tag{5}$$

$$\pi(C) = \frac{b}{n} \sum_{j} E_j - c \tag{6}$$

where b denotes a benefit from the public project and c is the cost of contribution. We assume that $c < b < nc$, so in the absence of punishment it is socially optimal for all members to contribute, while none of them have enough material motivation to do so. The term $\Pr(I_P)$, which we will specify precisely later, represents the probability with which punisher i would be injured or hurt during the punishment process; if this happens he pays the cost d. Similarly, defector i needs to pay s with the probability $\Pr(I_D)-$ the probability of the defector being injured. We assume that $s > c$, so the cost that the defector pays in case of being injured $-$ for example the cost of recovery from injury or foregone income from the exclusion of productive activities because of injury $-$ is greater than the per-period contribution cost. Note that when $d = s = 0$, no punishment takes place and payoffs replicate the n-prisoner dilemma.

Though we use the language of public goods problems, we observe that this setting can be readily extended to the situation of political collective actions (Tullock, 1971; DeNardo, 1985; Epstein, 2002). In the context of revolutionary activities to overthrow a corrupt and oppressive government, this setting models an individual's choice from among three activities: join the revolutionaries (punisher), join the forces of repression (defector), or remain inactive (cooperator). With these name changes (4)$\sim$(6) reproduce a similar specification of expected payoffs that Tullock (1971) used in his study of the "paradox of revolution".

We proceed to specify terms $\Pr(I_P)$ and $\Pr(I_C)$ using Lanchester's theory. First we introduce a parameter ρ to describe the degree of "collectivity" of the defectors in the punishment process:

$$\rho = \frac{1}{\text{the number of defectors who counteract together}}$$

Figure 2 illustrates this parameter schematically.

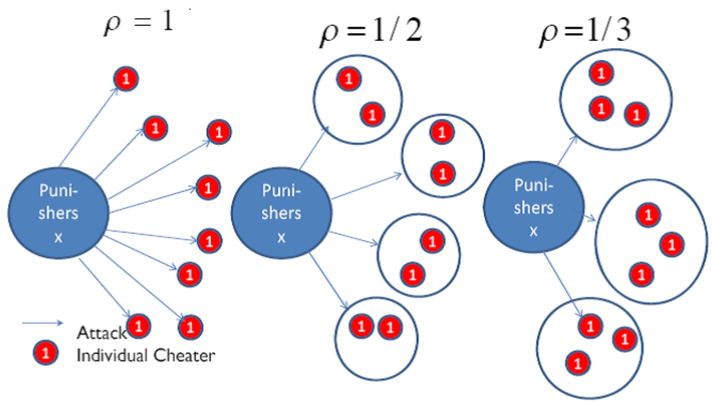

Figure 2: **Defectors' tendency to act "collectively" in the punishment.** $\frac{1}{\rho}$ represents the number of defectors who act together in the punishment process. When $\rho = 1$ all defectors behave individually in the punishing process and as $\rho \to 0$ defectors behave more collectively.

For instance, when the number of punishers and the defectors are x and $y = 2021$ and $\rho = 1/1000$, from a similar calculation the condition for the punishers to defeat "the army" of defectors is as follows:

$$x^2 \geq 2 \times (1000)^2 + 21^2 = [0.001 \times 2021] \times (\frac{1}{0.001})^2 + (2021 - 2000)^2$$

where $[x]$ denotes the integer part of x. The first term, $2 \times (1000)^2$, indicates that the army of punishers faces 1000 defectors twice and then competes with the remaining 21 defectors. Thus in general we have the following condition for the punishers to annihilate all defectors:

$$\phi x^2 \geq \kappa[\rho y] \left(\frac{1}{\rho}\right)^2 + \kappa(y - [\rho y]\frac{1}{\rho})^2 \tag{7}$$

where $\rho > 0$. Since a type is more likely to be injured or knocked out if the result of the punishment process is close to the defeat of that type, we suppose that $\Pr(I_D)$ and $\Pr(I_P)$ monotonically depends on (7). In particular, we suppose that

$$\begin{aligned}\Pr(I_D) &= F(\phi x^2 - \kappa[\rho y]\left(\frac{1}{\rho}\right)^2 - \kappa(y - [\rho y]\frac{1}{\rho})^2) \\ \Pr(I_P) &= 1 - F(\phi x^2 - \kappa[\rho y]\left(\frac{1}{\rho}\right)^2 - \kappa(y - [\rho y]\frac{1}{\rho})^2)\end{aligned}$$

where $F(t)$ increasing, $\lim\limits_{t\to\infty} F(t) = 1$, $\lim\limits_{t\to-\infty} F(t) = 0$. For example, $F(t) = \mathbf{1}_{[0,\infty)}(t)$, or $F(t) = 1$ if $t \geq 0$, $= \exp(\beta t)$ if $t < 0$.

3.2 Static Analysis

Using the model we have developed, we ask two questions: 1. How does an increase in group size change π_P, π_D, and π_C at a given time? 2. How does an increase in group size affect, at equilibrium, the proportion of each type in the population when individuals update their types? The first question addresses the static characterization of the model, while the second one concerns equilibrium states in the dynamics of the model. Of course, these two are closely related as the standard result in game theory suggests $-$ for instance, the strict Nash equilibrium in the underlying game is an evolutionarily stable strategy, and hence the asymptotically stable state in the replicator dynamics (See Weibull, 1995). Concerning the first question, we have proposition 1.

Proposition 1 *Suppose that $\alpha = \frac{x}{n}, \beta = \frac{y}{n}$, and $s > c$. Then for all $\rho > 0$, $\alpha > 0, \beta > 0$, $\lim_{n\to\infty}(\pi_P - \pi_D) > 0$ and $\lim_{n\to\infty}(\pi_C - \pi_D) > 0$*

Proof. From the definition of $\Pr(I_D)$ we have

$$\begin{aligned}\Pr(I_D) &= F(\phi\alpha^2 n^2 - \kappa[\rho\beta n]\left(\frac{1}{\rho}\right)^2 - \kappa(n\beta - [\rho n\beta]\frac{1}{\rho})^2) \\ &\to 1 \text{ as } n \to \infty\end{aligned}$$

since $\left|-\kappa(n\beta - [\rho n\beta]\frac{1}{\rho})^2\right| = \frac{\kappa}{\rho^2}\left|(\rho n\beta - [\rho n\beta])^2\right| \leq \frac{\kappa}{\rho^2}$. Then $\pi_C - \pi_D \geq \pi_P - \pi_D \to -c + s > 0$. ■

Since $\pi_C > \pi_D$, whenever $\pi_P > \pi_D$ holds, playing D is strictly dominated by both strategies C and P. Figure 3 below characterizes the combinations of the population proportion $(\alpha, \beta, 1-\alpha-\beta)$ which support $\pi_P > \pi_D$.

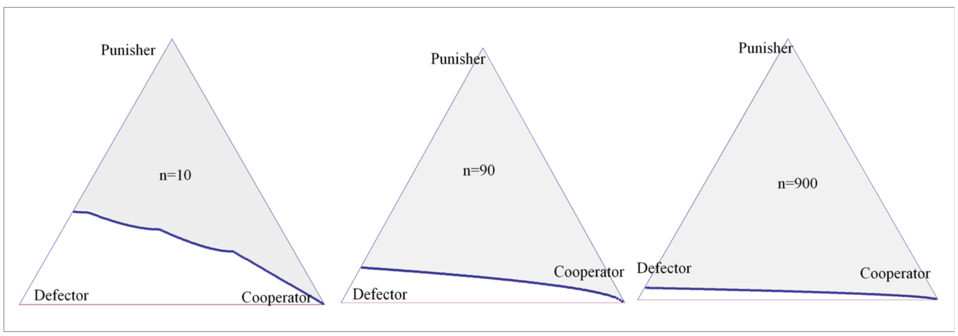

Figure 3: **The fractions of population which support punishment.** Each point in the triangles uniquely corresponds to one population state, composed of fractions of each type, through the Bary centric coordinate. For example, the point located on the left bottom vertex corresponds to a population state in which all individuals choose the defector type. The points in the shaded area are population states which ensure $\Pr(I_P) = 1$, so $\pi_P - \pi_D > 0$,so in the shaded region playing defector is strictly dominated. $\rho = 0.5, \kappa = \phi, F(t) = 1_{[0,\infty)}$ are used.

In each panel of figure 3 the shaded regions show the population state where playing defector is strictly dominated by punishers, and hence by cooperators. In contrast, when x, y belong to the unshaded regions, we have $\Pr(I_P) \approx 1$, $\Pr(I_D) \approx 0$, so $\pi_D > \pi_P$ and $\pi_D > \pi_C$ and playing defector is individually rational. When the size of the group increases, the shaded region enlarges; being defector becomes less favorable. In addition, because of these features of the payoffs we may regard the shaded regions, in a suitable dynamic process, as basins of attraction for some equilibrium supporting cooperation; unshaded regions may correspond to the basins of attraction for an all-defectors equilibrium. The static analysis of the payoff structures and the basins of attraction strongly suggests that cooperation would be supported in the long run.

3.3 Dynamic Analysis

We consider a state space $\Xi_n = \{(x, y, z) \in \mathbb{R}^3 |\ x + y + z = n\}$,which describes the numbers of punishers, defectors, and cooperators. Given that the state at the end of period of t is (x, y, z), we write $\pi_k(x, y) := \pi_k$, for $k = P, D, C$ to emphasize the dependence of payoffs on x, y. During the period $t + 1$

D1 A proportion of individuals is drawn from the population at random.

D2 With probability $(1-\epsilon)$ for $\epsilon \in (0, 1)$, the drawn individuals choose types according to the following switching rule:

$$\text{type } i \text{ switch to type } j \text{ if } j \in \arg\max_k \pi_k(x, y)$$

Whenever target strategies are more than one, an individual is assumed to choose one randomly.

D3 With probability ϵ, individuals choose types randomly and the system moves into the next period.

D2 is called a best response update (Kandori et al., 1993; Young, 1998) and the specification of stochastic dynamics follows Young (1998) except for D1. Instead of drawing one individual as Young (1998) does, we draw a proportion of individuals. If we draw one individual at each period, the convergence speed of the system to an equilibrium may slow down as n increases, so some positive level of punishing and cooperating behaviors may persist simply due to the sampling method. Since we wish to control this artifact and single out the large group effect from the irrelevant aspects of the modeling, we allow a proportion of individuals to update their strategies. In D3, we allow for the possibility of idiosyncratic behavior such as mistakes by individuals in choosing their best response strategies following the standard evolutionary model (Kandori et al., 1993; Young, 1998).

Since the independent randomness, which arises each period both by D1 and D3, accumulates in the system through time, the process follows a Markov chain and the standard limit theorem for finite state Markov processes applies. In particular, D3 makes the chain irreducible and aperiodic, so we have a unique invariant distribution μ. Since we are interested in the long run equilibrium value of population fractions, playing cooperators, punishers, and defectors, we estimate $\lim_{t\to\infty} E(X_t)$, $\lim_{t\to\infty} E(Y_t)$, and $\lim_{t\to\infty} E(Z_t)$ using a Monte Carlo simulation (Madras, 2002). As we do not know an invariant distribution we take the all-defectors state as an initial state, which is the least likely state to support a high level of cooperation in the long run. Figure 4 depicts trajectories of the states of the system. In the first panel, where the size of the population is relatively small, the population state starting from all defectors stays close to the all-defectors equilibrium. This may capture the situation in which all individuals are trapped in the basin of attraction of the all-defectors

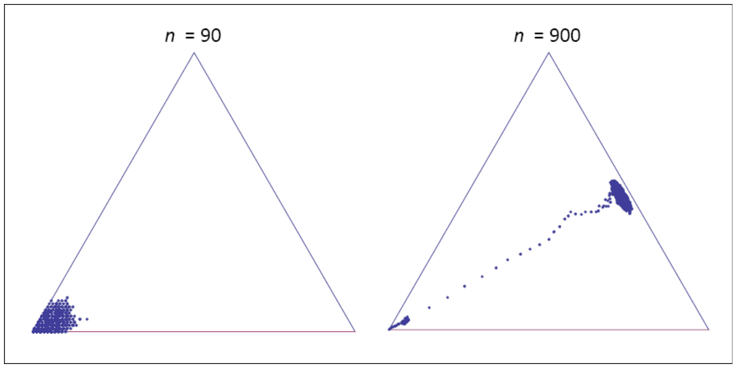

Figure 4: **The fractions of population in each period.** Each point is each state in the simulation. The initial values of states are taken as $\alpha = 0$, $\beta = 1$, $\gamma = 0$. 10% of population are drawn at each period. Other parameters used are $\kappa = 1, \phi = 1, \rho = 0.5, \epsilon = 0.1, \beta = 10, c = 2, d = 3, s = 3, T = 10000$.

equilibrium in figure 3. The second panel shows that as the basin of attraction for the all-defectors equilibrium shrinks, the system manages to escape from this equilibrium and reach the state with higher levels of cooperation. The estimates in table 1 corroborate the large group effect more precisely. As the size of the population grows, it becomes easier to prevent the proliferation of defectors with a smaller fraction of punishers.

	$n = 90$			$n = 900$		
	Punisher	Defector	Cooperator	Punisher	Defector	Cooperator
fraction	**0.03362**	**0.93242**	**0.03395**	**0.48231**	**0.033359**	**0.484325**
95% Conf.	[0.031855	[0.929807	[0.032531	[0.480872	[0.03275	[0.48274
Interval	0.0353861]	0.935051]	0.035368]	0.48376]	0.033960]	0.48591]

Table 1: **Estimates of mean fractions of population.** Estimates of the mean fractions of population and 95% confidence intervals. To estimate the mean fractions and construct the confidence intervals, we follow the batch means method (See Madras, 2002) and choose 25 batches. Also to avoid the initialization bias the first five batches have been dropped. The parameters used are the same as in figure 2.

4 Discussion

We note two features of the model. When $\rho \to 0$, the above argument fails to hold in the limit. This is because if $\rho \to 0$, meaning the defectors behave as collectively as the punishers do, the punisher cannot exploit Lanchester's law. However, proposition 1 does hold for all $\rho > 0$. As long as ρ remains positive (even if ρ is very close to 0) or the defector tends to behave less collectively, the punisher will always enjoy large group advantages. In addition, the result does not depend on the magnitude of d. This means that, however high the cost of punishment is, an increase in group size is always favors punishers and cooperators. This fact suggests that the second-order free rider problem can be reduced by the size of the group (Panchanathan and Boyd, 2004).

We have shown that if the punishment process is well described by Lanchester's equations, larger groups may favor cooperation. Of course this argument does not assert that larger groups are always successful in collective action; larger groups may have other disadvantages – for instance, higher coordination costs and information costs. However, by providing one instance of large-group advantages we verify that the Olsonian view of collective action and group size does not always provide a correct answer. Moreover, an appropriately modified theory, for instance one incorporating coordination costs, would provide better understanding of interesting questions of group and collective action such as the determination of the optimal size of a group.

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