Notation for Mixed Models for Finite Populations

Simple Population

Units and Response

$s = 1,\ldots,N$	Unit Labels
y_s for $s = 1,\ldots,N$	Expected Response (over replicated measurements)
x_s for $s = 1,\ldots,N$	Regression variables (Luz Mery)
x_{ks} for $s = 1,\ldots,N$, $k = 1,\ldots,p$	Auxiliary variables for unit s (Wenjun)
$z_{ks} = x_{ks} - \mu_k$ for $s = 1,\ldots,N$, $k = 1,\ldots,p$	Auxiliary variable deviation from mean for unit s
z_{0s} for $s = 1,\ldots,N$	Expected Response (over replicated measurements)(notation used with auxiliary information) (Wenjun)
w_s for $s = 1,\ldots,N$ where $\sum_{s=1}^{N} w_s = 1$	Weight for unit s.
$w_s = \frac{(x_s - \mu_x)^2}{\sum_{s=1}^{N}(x_s - \mu_x)^2}$	Weight for regression parameter (Luz Mery)
$y_{ws} = w_s y_s$ for $s = 1,\ldots,N$	Weighed expected response for unit s
$x_s^{\Diamond} = \frac{x_s - \mu_x}{(N-1)\sigma_x^2}$ for $s = 1,\ldots,N$	Regression approach (Luz Mery)
$y_s^{\Diamond} = y_s x_s^{\Diamond}$ for $s = 1,\ldots,N$	Regression approach (Luz Mery)
$e_s = y_s - (A + Bx_s)$	Lack of fit in regression approach (Luz Mery)
$b_s = \frac{y_s - \mu_y}{x_s - \mu_x}$	Slope for unit s based on connecting population means (Luz Mery)

With Measurement Error

$k = 1,\ldots,r_s$	Index for replication
W_{sk} or W_s (if $r_s = 1$) for $k = 1,\ldots,r_s$	Measurement error for k^{th} replication of unit s

$\tilde{W}_{ik}$ or $\tilde{W}_i$ (if $r_s = 1$) for $k = 1,...,r_s$	Measurement error for k^{th} replication on the unit in position i
$Y_{sk} = y_s + W_{sk}$ for $k = 1,...,r_s$	Measurement for k^{th} replication.
$Y_{wsk} = w_s Y_{sk}$	Weighted response for k^{th} replication of unit s

Conventions:

Use the subscript R for vectors when the only random component is measurement error.

Use the subscript pos for expectation over position measurement error (Luz Mery)

Use the subscript ξ_3 for expectation over measurement error on units in clusters.

Use the subscript ξ_2 for expectation with respect to permutations of units in a cluster.

Use the subscript ξ_1 for expectation with respect to permutations of the clusters.

Use the subscript ξ for expectation with respect to permutations of the units of a simple population.

Use the subscript S for expectation with respect to permutations of the units of a simple population (Luz Mery).

Use a super-script * for vectors when units and measurement error are random.

Use an over-script $\overleftrightarrow{Y}_{is} = U_{is} y_s$ when random variables are expanded.

<u>Vectors</u>

$\mathbf{y} = ((y_s)) = (y_1 \quad y_2 \quad \cdots \quad y_N)'$	Expected response vector (with respect to measurement)
$\mathbf{x}_s = ((x_{ks})) = (x_{1s} \quad x_{2s} \quad \cdots \quad x_{ps})'$	Auxiliary response for unit s (Wenjun)
$\mathbf{x} = ((x_s)) = (x_1 \quad x_2 \quad \cdots \quad x_N)'$	Auxiliary response (Luz Mery)
$\mathbf{x}^{\Diamond} = ((x_s^{\Diamond})) = (x_1^{\Diamond} \quad x_2^{\Diamond} \quad \cdots \quad x_N^{\Diamond})'$	Auxiliary response (Luz Mery)
$\mathbf{z}_s = ((z_{ks})) = (z_{0s} \quad z_{1s} \quad \cdots \quad z_{ps})'$	
$\mathbf{z} = ((\mathbf{z}_s'))$	
$\mathbf{w} = ((w_s)) = (w_1 \quad w_2 \quad \cdots \quad w_N)'$	Vector of weights

$\mathbf{y}_w = ((y_{ws})) = (y_{w1} \quad y_{w2} \quad \cdots \quad y_{wN})'$ Expected weighted response vector (with respect to measurement)

$\mathbf{Y}_R = ((Y_{sk})) = (Y_{1k} \quad Y_{2k} \quad \cdots \quad Y_{Nk})'$ Measured response vector (for $r_s = 1$ for all $s = 1,...,N$),

$\mathbf{Y}_{wR} = ((Y_{wsk})) = (Y_{w1k} \quad Y_{w2k} \quad \cdots \quad Y_{wNk})'$ Measured weighted response vector (for $r_s = 1$ for all $s = 1,...,N$),

$\mathbf{W} = ((W_{sk})) = (W_{1k} \quad W_{2k} \quad \cdots \quad W_{Nk})'$ Measurement error vector (for $r_s = 1$ for all $s = 1,...,N$),

$\boldsymbol{\alpha}$ General vector of parameters.

$\mathbf{e}_i$ Constant vector with 1 in position i and zero elsewhere

Parameters and Constants

$\bar{c}_{II} = \frac{1}{N-n}\sum_{i=n+1}^{N} c_i$ Average remaining coefficient when predicting a mean using auxiliary variables

$\sigma^2_{C_{II}} = \frac{1}{N-n-1}\sum_{i=n+1}^{N}\left(c_i - \bar{c}_{II}\right)^2$ Used in EMSE of Regression Predictor (Luz Mery)

$f = \frac{n}{N}$ Sampling fraction (Wenjun) (but this may be defined differently in 2-stage sampling)

$\mu = \frac{1}{N}\sum_{s=1}^{N} y_s$ Usual definition

$\mu_y = \frac{1}{N}\sum_{s=1}^{N} y_s$ Definition used with auxiliary variables, regression.

$\mu_{y^\Diamond} = \frac{1}{N}\sum_{s=1}^{N} y_s^\Diamond$ Regression problem (Luz Mery)

$\mu_x = \frac{1}{N}\sum_{s=1}^{N} x_s$ Definition used with single regression problem

$\mu_w = \frac{1}{N}\sum_{s=1}^{N} w_s y_s$

$A = \mu_y - B\mu_x$ Regression intercept

$$B=\frac{\sum_{s=1}^{N}\left(y_s-\mu_y\right)\left(x_s-\mu_x\right)}{\sum_{s=1}^{N}\left(x_s-\mu_x\right)^2}$$ Regression parameter for association. (Luz Mery)

$$\sigma^2=\frac{1}{N-1}\sum_{s=1}^{N}\left(y_s-\mu\right)^2$$ or

$$\sigma_y^2=\frac{1}{N-1}\sum_{s=1}^{N}\left(y_s-\mu_y\right)^2$$ Regression problem (Luz Mery)

$$\sigma_x^2=\frac{1}{N-1}\sum_{s=1}^{N}\left(x_s-\mu_x\right)^2$$ Regression problem (Luz Mery)

$$\sigma_{y^\Diamond}^2=\frac{1}{N-1}\sum_{s=1}^{N}\left(y_s^\Diamond-\mu_{y^\Diamond}\right)^2$$ Regression problem (Luz Mery)

$$\sigma_w^2=\frac{1}{N-1}\sum_{s=1}^{N}\left(w_s y_s-\mu_w\right)^2$$

$$\sigma_{xy}=\frac{1}{N-1}\sum_{s=1}^{N}\left(y_s-\mu_y\right)\left(x_s-\mu_x\right)$$ Regression (Luz Mery)

$$\sigma_{y^\Diamond x}=\frac{1}{N-1}\sum_{s=1}^{N}\left(y_s^\Diamond-\mu_{y^\Diamond}\right)\left(x_s-\mu_x\right)$$

$$\rho_{xy}=\frac{\sigma_{xy}}{\sigma_x\sigma_y}$$

$\text{var}_R\left(W_{sk}\right)=\sigma_{se}^2$ Measurement error variance associated with unit s

$\text{var}_R\left(\tilde{W}_{ik}\right)=\sigma_{iL}^2$ Measurement error variance (e.g. interviewer, laboratory) associated with position i (Ed)

$\text{var}_{pos}\left(\tilde{W}_i\right)=\tilde{\sigma}_i^2$ Measurement error variance associated with position i (Luz Mery)

$$\sigma_e^2=\frac{1}{N}\sum_{s=1}^{N}\sigma_{se}^2$$ (Ed)

$$\bar{\sigma}_e^2=\frac{1}{N}\sum_{s=1}^{N}\sigma_{se}^2$$ (Luz Mery)

$\bar{\sigma}_e^{\Diamond 2} = \frac{1}{N}\sum_{s=1}^{N} x_s^{\Diamond 2}\sigma_{se}^2$ (Luz Mery)

$\sigma_{we}^2 = \frac{1}{N}\sum_{s=1}^{N} w_s^2\sigma_{se}^2$

$k = \frac{\sigma^2}{\sigma^2 + \sigma_e^2}$ Random permutation shrinkage constant for simple random sampling with unit measurement error where $r_s = 1$ for all $s = 1,...,N$.

$\tilde{k}_i = \frac{\sigma^2}{\sigma^2 + \sigma_{iL}^2}$ Random permutation shrinkage constant for simple random sampling with position measurement error where $r_s = 1$ for all $s = 1,...,N$ (Ed).

$k_i^{+} = \frac{\sigma^2}{\sigma^2 + \sigma_i^2}$ Partially conditional random permutation shrinkage constant for simple random sampling with position measurement error where $r_s = 1$ for all $s = 1,...,N$ (Ed).

$k^{+} = \frac{\sigma_y^2\left(1-\rho_{xy}^2\right)}{\bar{\sigma}_e^2 + \sigma_y^2\left(1-\rho_{xy}^2\right)}$ Constant for Reg with unit measurement error (Luz Mery)

$k^{\Diamond +} = \frac{\sigma_{y^\Diamond}^2\left(1-\rho_{xy^\Diamond}^2\right)}{\bar{\sigma}_e^{\Diamond 2} + \sigma_{y^\Diamond}^2\left(1-\rho_{xy^\Diamond}^2\right)}$ Regression problem (Luz Mery)

$\tilde{k}_i = \frac{\sigma_y^2\left(1-\rho_{xy}^2\right)}{\tilde{\sigma}_i^2 + \sigma_y^2\left(1-\rho_{xy}^2\right)}$ Constant for Reg with Position error (Luz Mery)

$\tilde{k}_i^{*} = \frac{\tilde{k}_i}{\sum_{i=1}^{n}\tilde{k}_i}$ Used by Luz Mery for Regression

$D^{+} = \left(N-n\right)\bar{c}_{II} + n\left(1+k^{+}\right)\bar{c}_I$ Constant for Regression with measurement error (Luz Mery)

$\tilde{D} = \left(N-n\right)\bar{c}_{II} + \sum_{i=1}^{n} c_i\left(1+\tilde{k}_i\right)$ Constant for Regression with position error (Luz Mery)

$\sigma_i^2 = \mathbf{u}_i' \left(\overset{N}{\underset{s=1}{\oplus}} \sigma_{se}^2 \right) \mathbf{u}_i$ Measurement error variance associated with the unit realized in position i.

(Notice that we could define $\sigma_i^2 = \mathbf{u}_i' \left(\overset{N}{\underset{s=1}{\oplus}} \sigma_{se}^2 \right) \mathbf{u}_i + \sigma_{iL}^2$, which seems better.)

Vectors and Matrices

$$\mathbf{I}_a = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix};$$

$$\mathbf{1}_a = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix};$$

$$\mathbf{J}_a = \mathbf{1}_a \mathbf{1}_a' = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix};$$

$$\mathbf{D}_{\mathbf{a}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_c \end{pmatrix}$$ where $\mathbf{a} = \left(\left(a_r \right) \right) = \left(a_1 \quad a_2 \quad \cdots \quad a_c \right)'$. A diagonal matrix.

$\mathbf{D}_{\bar{\sigma}_s^2}$ an $N \times N$ diagonal matrix with elements $\bar{\sigma}_s^2$ on the diagonal.

$$\mathbf{P}_a = \mathbf{I}_a - \frac{1}{a} \mathbf{J}_a$$

$$\mathbf{P}_{a,b} = \mathbf{I}_a - \frac{1}{b} \mathbf{J}_a$$

$\mathbf{R} = \begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_N \end{pmatrix}$ Used to subtract mean in regression (Luz Mery)

$\mathbf{\Delta} = \frac{1}{N-1} \left(\overset{N}{\underset{s=1}{\oplus}} y_s \right) \mathbf{P}_N \left(\overset{N}{\underset{s=1}{\oplus}} y_s \right)$ where we note that $\sigma^2 = \mathbf{1}_N' \mathbf{\Delta} \mathbf{1}_N$ and $\sigma_w^2 = \mathbf{w}' \mathbf{\Delta} \mathbf{w}$

$\mathbf{L} = \begin{pmatrix} \mathbf{L}_I \\ \mathbf{L}_{II} \end{pmatrix}$ used to collapse an partition random variables, such as $\begin{pmatrix} \mathbf{L}_I \\ \mathbf{L}_{II} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbf{I}_n & \underset{n \times N-n}{\mathbf{0}} \end{pmatrix} \otimes \mathbf{1}_N' \\ \hdashline \begin{pmatrix} \underset{(N-n)xn}{\mathbf{0}} & \mathbf{I}_{N-n} \end{pmatrix} \otimes \mathbf{1}_N' \end{pmatrix}$

$\mathbf{K}=\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}$ used to partition random variables into samples, remainder, such that

$$\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}=\begin{pmatrix}\mathbf{I}_2\otimes\begin{pmatrix}\mathbf{I}_n & \mathbf{0}_{n\times N-n}\end{pmatrix}\\ \hline \mathbf{I}_2\otimes\begin{pmatrix}\mathbf{0}_{(N-n)xn} & \mathbf{I}_{N-n}\end{pmatrix}\end{pmatrix}$$ in regression (Luz Mery)

$\mathbf{g}'=\begin{pmatrix}\mathbf{g}'_I & \mathbf{g}'_{II}\end{pmatrix}$ constants to determine 'target'

$\mathbf{c}=\begin{pmatrix}\mathbf{c}'_I & \mathbf{c}'_{II}\end{pmatrix}'$ Constants to determine target with auxiliary variables (Wenjun).

$\mathbf{c}_I=\left(\left(c_i\right)\right)=\begin{pmatrix}c_1 & c_2 & \cdots & c_n\end{pmatrix}'$ Partitioned constants for sample (Wenjun)

$\mathbf{C}_I=\left(\left(c_i\right)\right)=\begin{pmatrix}c_1 & c_2 & \cdots & c_n\end{pmatrix}'$ Partitioned Constants to define regression target (Luz Mery)

$\mathbf{c}_{II}=\left(\left(c_{n+i}\right)\right)=\begin{pmatrix}c_{n+1} & c_{n+2} & \cdots & c_N\end{pmatrix}'$ Partitioned constants for remainder (Wenjun)

$\mathbf{C}_{II}=\left(\left(c_{n+i}\right)\right)=\begin{pmatrix}c_{n+1} & c_{n+2} & \cdots & c_N\end{pmatrix}'$ Partitioned Constants to define regression target (Luz Mery)

$\mathbf{C}_I=\begin{pmatrix}\mathbf{c}'_I & \mathbf{0}'_{np\times 1}\end{pmatrix}'$ Partitioned linear combination for sample with auxiliary variables (Wenjun)

$\mathbf{C}_{II}=\begin{pmatrix}\mathbf{c}'_{II} & \mathbf{0}'_{(N-n)p\times 1}\end{pmatrix}'$ Partitioned linear combination for remaining with auxiliary variables (Wenjun)

$\mathbf{e}'_i=\begin{pmatrix}\mathbf{e}'_{iI} & \mathbf{e}'_{iII}\end{pmatrix}$ Partitioned version of indicator vector for position i.

$\mathbf{G}_I=\begin{pmatrix}\mathbf{1}'_n & \mathbf{0}'_{np}\end{pmatrix}'$ Expression similar to design matrix with auxiliary sample random variables

$\mathbf{g}_I=\begin{pmatrix}\mathbf{C}'_I & \mathbf{0}'_{n\times 1}\end{pmatrix}'$ Partitioned linear combination for sample with regression (Luz Mery)

$\mathbf{G}_{II}=\begin{pmatrix}\mathbf{1}'_{N-n} & \mathbf{0}'_{(N-n)p}\end{pmatrix}'$ Expression similar to design matrix for remainder random variables.

$\mathbf{G}'_S=\mathbf{G}'\mathbf{K}'_S$ Sample linear combination to define target by Viviana

$\mathbf{G}'_R=\mathbf{G}'\mathbf{K}'_R$ Remainder linear combination to define target by Viviana

$\boldsymbol{\mu}_x=\left(\left(\mu_k\right)\right)=\frac{1}{N}\sum_{s=1}^{N}\mathbf{x}_s$ (Wenjun)

$\boldsymbol{\mu}_z=\begin{pmatrix}\mu_y & \mathbf{0}'_p\end{pmatrix}'$ (Wenjun)

$\boldsymbol{\mu} = \begin{pmatrix} \mu_y & \mu_x \end{pmatrix}'$ (Luz Mery)

$$\boldsymbol{\Sigma} = \frac{1}{N-1}\sum_{s=1}^{N}\left(\mathbf{z}_s - \boldsymbol{\mu}_z\right)\left(\mathbf{z}_s - \boldsymbol{\mu}_z\right)' = \begin{pmatrix} \sigma_y^2 & \boldsymbol{\sigma}'_{yX} \\ \boldsymbol{\sigma}_{yX} & \boldsymbol{\Sigma}_X \end{pmatrix}$$

$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_y^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_x^2 \end{pmatrix}$ Variance matrix (Luz Mery)

$\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} \sigma_{11}^{\circ} & \sigma_{12}^{\circ} \\ \sigma_{21}^{\circ} & \sigma_{22}^{\circ} \end{pmatrix}$ Inverse variance for regression used by Luz Mery

$\boldsymbol{\sigma}_{yX} = \begin{pmatrix} \sigma_{yx_1} & \sigma_{yx_2} & \cdots & \sigma_{yx_p} \end{pmatrix}'$

$\boldsymbol{\Sigma}_X = \left(\left(\sigma_{x_k x_{k^*}}\right)\right)$

$\rho_{yX}^2 = \boldsymbol{\sigma}'_{Xy}\boldsymbol{\Sigma}_X^{-1}\boldsymbol{\sigma}_{Xy}/\sigma_y^2$ Squared multiple correlation coefficient of Y on $\mathbf{X}$

$\boldsymbol{\beta} = \boldsymbol{\Sigma}_X^{-1}\boldsymbol{\sigma}_{yX} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_p \end{pmatrix}'$ Regression parameters between response and auxiliary variables

$\beta_s = y_s - \mu$, so that $y_s = \mu + \beta_s$ and $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_N \end{pmatrix}'$ (Double definitions!).

Permutation Random Variables-

$i = 1,\ldots,N$ Position in the permutation.

U_{is} for $i = 1,\ldots,N$; $s = 1,\ldots,N$ Indicator of selection of unit s in position i.

u_{is} for $i = 1,\ldots,N$; $s = 1,\ldots,N$ Realization of the random variable U_{is} for a permutation

$S_i = \sum_{s=1}^{N} U_{is} s$ for $i = 1,\ldots,N$ Random variable representing label for unit in position i

$Y_i = \sum_{s=1}^{N} U_{is} y_s$ for $i = 1,\ldots,N$ Random variable representing expected response for unit in position i

$Y_{wi} = \sum_{s=1}^{N} U_{is} y_{ws}$ for $i = 1,\ldots,N$ Random variable representing expected weighted response

$X_i = \sum_{s=1}^{N} U_{is} x_s$ for $i = 1,\ldots,N$ Regression approach (Luz Mery)

$X_i^{\lozenge} = \dfrac{X_i - \mu_x}{(N-1)\sigma_x^2}$ Regression approach (Luz Mery $\lozenge$ =star)

$Y_i^{\lozenge} = Y_i X_i^{\lozenge}$ Regression approach (Luz Mery $\lozenge$ =star)

$Y_i^{+} = \sum_{s=1}^{N} U_{is}\left(y_s + W_s\right)$ Response for the unit in position i (for $r_s = 1$ for all $s = 1,\dots,N$) when there is measurement error (Luz Mery).

$Y_i^{+} = Y_i + \sum_{s=1}^{N} u_{is} W_{sk}$ or $Y_i^{+} = Y_i + \tilde{W}_{ik}$ Partially conditional response (Ed) (Double definitions!)

$Y_i^{\lozenge +} = Y_i^{+} X_i^{\lozenge}$ Regression approach (Luz Mery $\lozenge$ =star)

$Y_{ik}^{*} = \sum_{s=1}^{N} U_{is} Y_{sk}$ or Y_i^{*} Response for the unit in position i (for $r_s = 1$ for all $s = 1,\dots,N$) when there is measurement error (Ed).

$Y_{wik}^{*} = \sum_{s=1}^{N} U_{is} Y_{wsk}$

$\tilde{Y}_{ik} = \sum_{s=1}^{N} U_{is} y_s + \tilde{W}_{ik}$ or $\tilde{Y}_i$ (when $k = 1$) Response for the unit in position i when there is position (i.e. interviewer) error

$W_{ik}^{*} = \sum_{s=1}^{N} U_{is} W_{sk}$ or $W_{ik}^{*} = \mathbf{U}_i'\mathbf{W}$ Replication error for the unit in position i (for $r_s = 1$ for all $s = 1,\dots,N$)

$\tilde{W}_{ik}$ or $\tilde{W}_i$ (if $r_s = 1$) Measurement error associated with Position (i.e. interviewer) for the unit in position i.

$\tilde{W}_i^{\lozenge} = \tilde{W}_i X_i^{\lozenge}$ Regression problem (Luz Mery)

$\mathbf{U}_i = \left(\left(U_{is}\right)\right) = \begin{pmatrix} U_{i1} & U_{i2} & \cdots & U_{iN} \end{pmatrix}'$

$\mathbf{u}_i = \left(\left(u_{is}\right)\right) = \begin{pmatrix} u_{i1} & u_{i2} & \cdots & u_{iN} \end{pmatrix}'$ Realization of the vector $\mathbf{U}_i$.

$\mathbf{U} = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 & \cdots & \mathbf{U}_N \end{pmatrix}'$

Expanded Random Variables

$\overleftrightarrow{Y}_{is} = U_{is} y_s$ for $i = 1,...,N$ and $s = 1,...,N$ Random variable representing for unit s and position i in a permutation.

$\overleftrightarrow{Y}_{isk} = U_{is} Y_{sk}$ for $i = 1,...,N$ and $s = 1,...,N$ Random variable representing response for unit s and position i in a permutation.

$\overleftrightarrow{W}_{isk} = U_{is} W_{sk}$ for $i = 1,...,N$ and $s = 1,...,N$ Random variable representing measurement error for k^{th} replication for unit s and position i in a permutation.

$\overleftrightarrow{\mathbf{Y}}_i = \left(\left(U_{is} y_s \right) \right) = \begin{pmatrix} U_{i1} y_1 & U_{i2} y_2 & \cdots & U_{iN} y_N \end{pmatrix}'$ or $\overleftrightarrow{\mathbf{Y}}_i = \left(\bigoplus_{s=1}^{N} y_s \right) \mathbf{U}_i$

$\overleftrightarrow{\mathbf{Y}}_i^* = \left(\left(U_{is} \left(y_s + W_{sk} \right) \right) \right) = \begin{pmatrix} U_{i1} \left(y_1 + W_{1k} \right) & U_{i2} \left(y_2 + W_{2k} \right) & \cdots & U_{iN} \left(y_N + W_{Nk} \right) \end{pmatrix}'$

or $\overleftrightarrow{\mathbf{Y}}_i^* = \overleftrightarrow{\mathbf{Y}}_i + \overleftrightarrow{\mathbf{W}}_i = \left(\bigoplus_{s=1}^{N} \left(y_s + W_{sk} \right) \right) \mathbf{U}_i$ where $r_s = 1$ for all $s = 1,...,N$

$\overleftrightarrow{\mathbf{Y}}^* = vec\begin{pmatrix} \overleftrightarrow{\mathbf{Y}}_1^* & \overleftrightarrow{\mathbf{Y}}_2^* & \cdots & \overleftrightarrow{\mathbf{Y}}_N^* \end{pmatrix}$ or $\overleftrightarrow{\mathbf{Y}}^* = vec\left[\left(\bigoplus_{s=1}^{N} \left(y_s + W_{sk} \right) \right) \mathbf{U}' \right]$

$\overleftrightarrow{\mathbf{Y}} = vec\begin{pmatrix} \overleftrightarrow{\mathbf{Y}}_1 & \overleftrightarrow{\mathbf{Y}}_2 & \cdots & \overleftrightarrow{\mathbf{Y}}_N \end{pmatrix}$ or $\overleftrightarrow{\mathbf{Y}} = vec\left[\left(\bigoplus_{s=1}^{N} y_s \right) \mathbf{U}' \right]$

$\overleftrightarrow{\mathbf{W}}_i = \left(\left(U_{is} W_{sk} \right) \right) = \begin{pmatrix} U_{i1} W_{1k} & U_{i2} W_{2k} & \cdots & U_{iN} W_{Nk} \end{pmatrix}'$ or $\overleftrightarrow{\mathbf{W}}_i = \left(\bigoplus_{s=1}^{N} W_{sk} \right) \mathbf{U}_i$ where $r_s = 1$ for all $s = 1,...,N$

Collapsed Random Variables

$\mathbf{L}$ used to collapse random variables, as for example in $\mathbf{L}\overleftrightarrow{\mathbf{Y}} = \left(\mathbf{A} \otimes \mathbf{B} \right) \overleftrightarrow{\mathbf{Y}}$

$\mathbf{Y} = \left(\left(Y_i \right) \right) = \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_N \end{pmatrix}'$ or $\mathbf{Y} = \mathbf{U}\mathbf{y}$

$\mathbf{Y}_w = \left(\left(Y_{wi} \right) \right) = \begin{pmatrix} Y_{w1} & Y_{w2} & \cdots & Y_{wN} \end{pmatrix}'$ or $\mathbf{Y} = \mathbf{U}\mathbf{y}$

$\mathbf{Y}^* = \left(\left(Y_{ik}^*\right)\right) = \begin{pmatrix} Y_{1k}^* & Y_{2k}^* & \cdots & Y_{Nk}^* \end{pmatrix}'$ or $\mathbf{Y}^* = \mathbf{Y} + \mathbf{W}^*$ when $r_s = 1$ for all $s = 1,...,N$ (Ed)

$\mathbf{Y}^+ = \left(\left(Y_i^+\right)\right) = \begin{pmatrix} Y_1^+ & Y_2^+ & \cdots & Y_N^+ \end{pmatrix}'$ or $\mathbf{Y}^+ = \mathbf{Y} + \mathbf{UW}$ when $r_s = 1$ for all $s = 1,...,N$ (Luz Mery)

$\tilde{\mathbf{Y}} = \mathbf{Y} + \tilde{\mathbf{W}}$ Regression problem with position error, (Luz Mery)

$\mathbf{Y}_w^* = \left(\left(Y_{wik}^*\right)\right) = \begin{pmatrix} Y_{w1k}^* & Y_{w2k}^* & \cdots & Y_{wNk}^* \end{pmatrix}'$ $\mathbf{Y}_w^* = \left(\left(Y_{wik}^*\right)\right) = \begin{pmatrix} Y_{w1k}^* & Y_{w2k}^* & \cdots & Y_{wNk}^* \end{pmatrix}'$

$\mathbf{W}^* = \left(\left(W_{ik}^*\right)\right) = \begin{pmatrix} W_{1k}^* & W_{2k}^* & \cdots & W_{Nk}^* \end{pmatrix}'$ or $\mathbf{W}^* = \mathbf{UW}$ when $r_s = 1$ for all $s = 1,...,N$

$\tilde{\mathbf{W}} = \left(\left(\tilde{W}_i\right)\right) = \begin{pmatrix} \tilde{W}_1 & \tilde{W}_2 & \cdots & \tilde{W}_N \end{pmatrix}'$ when $r_s = 1$ for all $s = 1,...,N$, position measurement error (Luz Mery)

$\mathbf{X} = \mathbf{Ux}$

$\mathbf{Z}_i = \sum_{s=1}^{N} U_{is}\mathbf{z}_s = \mathbf{z}'\mathbf{U}_i$

$\mathbf{Z} = \mathbf{Uz}$

$\mathbf{Z}^+ = vec\begin{pmatrix} \mathbf{Y}^+ & \mathbf{X} \end{pmatrix}$ Regression problem with measurement error (Luz Mery)

$\tilde{\mathbf{Z}} = vec\begin{pmatrix} \tilde{\mathbf{Y}} & \mathbf{X} \end{pmatrix}$ Regression problem with position error (Luz Mery)

$\mathbf{E}^+ = vec\begin{pmatrix} \mathbf{E}_y^+ & \mathbf{E}_x \end{pmatrix}$ Residual in regression (Luz Mery)

$\mathbf{E}_y^+ = \mathbf{Y}^+ - \mathbf{1}_N \mu_y$ Measurement error (unit) called Response Error (Luz Mery)

$\tilde{\mathbf{E}}_y = \tilde{\mathbf{Y}} - \mathbf{1}_N \mu_y$ Measurement error (position) (Luz Mery)

$\mathbf{E}_x = \mathbf{X} - \mathbf{1}_N \mu_x$ Deviations from mean (Luz Mery)

Sample and Remainder

$i = 1,...,n$ Sample

$Y_I = \sum_{i=1}^{n} Y_i$ Sample total of Expected Respons

$Y_{II} = \sum_{i=n+1}^{N} Y_i$ Remainder total

$Y_I^* = \sum_{i=1}^{n} Y_{ik}^*$ Sample total of Measured response where when $r_s = 1$ for all $s = 1,...,N$

$Y_{II}^* = \sum_{i=n+1}^{N} Y_{ik}^*$ Remainder total of Measured response where when $r_s = 1$ for all $s = 1,...,N$

$Y_{wI} = \sum_{i=1}^{n} Y_{wi}$ Sample total for weighted expected response

$Y_{wII} = \sum_{i=n+1}^{N} Y_{wi}$ Remainder total for weighted expected response

$Y_{wI}^* = \sum_{i=1}^{n} Y_{wik}^*$ Sample total weighted measured response when $r_s = 1$ for all $s = 1,...,N$

$Y_{wII}^* = \sum_{i=n+1}^{N} Y_{wik}^*$ Remainder total weighted measured response when $r_s = 1$ for all $s = 1,...,N$

$\bar{Y}_I = \frac{1}{n}\sum_{i=1}^{n} Y_i$ Sample average

$\bar{Y}_{II} = \frac{1}{N-n}\sum_{i=n+1}^{N} Y_i$ Remainder average

$\bar{X}_{kI} = \frac{1}{n}\sum_{i=1}^{n} X_{ki}$ Sample mean of auxiliary variable (Wenjun)

$\bar{X}_I = \frac{1}{n}\sum_{i=1}^{n} X_i$ Sample mean for one auxiliary variable (Luz Mery)

$\bar{Y}_I^* = \frac{1}{n}\sum_{i=1}^{n} Y_{ik}^*$ or $\bar{Y}^*$ Sample average of Measured response where when $r_s = 1$ for all $s = 1,...,N$

$\bar{Y}_{II}^* = \frac{1}{N-n}\sum_{i=n+1}^{N} Y_{ik}^*$ Remainder average of Measured response where when $r_s = 1$ for all $s = 1,...,N$

$\bar{Y}_I^+ = \frac{1}{n}\sum_{i=1}^{n} Y_i^+$ Sample mean for Regression predictor with unit measurement error (Luz Mery)

$\bar{Y}_I^{\Diamond +} = \frac{1}{n}\sum_{i=1}^{n} Y_i^{\Diamond +}$ Regression problem (Luz Mery)

Vectors

$$\mathbf{y}_I = ((y_s)) = (y_1 \quad y_2 \quad \cdots \quad y_n)'$$

$$\mathbf{y}_{II} = ((y_s)) = (y_{n+1} \quad y_{n+2} \quad \cdots \quad y_N)'$$

$$\mathbf{y}_{wI} = ((y_{ws})) = (y_{w1} \quad y_{w2} \quad \cdots \quad y_{wn})'$$

$$\mathbf{y}_{wII} = ((y_{ws})) = \left(y_{w(n+1)} \quad y_{w(n+2)} \quad \cdots \quad y_{wN}\right)'$$

$$\mathbf{Y}_{RI} = ((Y_{sk})) = (Y_{1k} \quad Y_{2k} \quad \cdots \quad Y_{nk})'$$

$$\mathbf{Y}_{RII} = ((Y_{sk})) = \left(Y_{(n+1)k} \quad Y_{(n+1)k} \quad \cdots \quad Y_{Nk}\right)'$$

$$\mathbf{Y}_{wRI} = ((Y_{wsk})) = (Y_{w1k} \quad Y_{wsk} \quad \cdots \quad Y_{wnk})'$$

$$\mathbf{Y}_{wRII} = ((Y_{wsk})) = \left(Y_{w(n+1)k} \quad Y_{w(n+2)k} \quad \cdots \quad Y_{wNk}\right)'$$

$$\mathbf{Y}_I = ((Y_i)) = (Y_1 \quad Y_2 \quad \cdots \quad Y_n)'$$

$\mathbf{Y}_S = ((Y_i)) = (Y_1 \quad Y_2 \quad \cdots \quad Y_n)'$ used by Viviana

$$\mathbf{Y}_{II} = ((Y_i)) = (Y_{n+1} \quad Y_{n+2} \quad \cdots \quad Y_N)'$$

$\mathbf{Y}_R = ((Y_i)) = (Y_{n+1} \quad Y_{n+2} \quad \cdots \quad Y_N)'$ used by Viviana

$\mathbf{Y}_I^* = ((Y_{ik}^*)) = (Y_{1k}^* \quad Y_{2k}^* \quad \cdots \quad Y_{nk}^*)'$ where $\mathbf{Y}_I^* = \left(\mathbf{I}_n \quad \underset{n\times N-n}{\mathbf{0}}\right)\mathbf{Y}^*$ when $r_s = 1$ for all $s = 1,\ldots,N$

$\mathbf{Y}_{II}^* = ((Y_{ik}^*)) = \left(Y_{(n+1)k}^* \quad Y_{(n+2)k}^* \quad \cdots \quad Y_{Nk}^*\right)'$ where $\mathbf{Y}_{II}^* = \left(\underset{(N-n)xn}{\mathbf{0}} \quad \mathbf{I}_{N-n}\right)\mathbf{Y}^*$ when $r_s = 1$ for all $s = 1,\ldots,N$

$$\mathbf{Y}_{wI} = ((Y_{wi})) = (Y_{w1} \quad Y_{w2} \quad \cdots \quad Y_{wn})'$$

$$\mathbf{Y}_{wII} = ((Y_{wi})) = \left(Y_{w(n+1)} \quad Y_{w(n+2)} \quad \cdots \quad Y_{wN}\right)'$$

$$\mathbf{Y}_{wI}^* = ((Y_{wik}^*)) = \left(Y_{w1k}^* \quad Y_{w2k}^* \quad \cdots \quad Y_{wnk}^*\right)'$$

$$\mathbf{Y}_{wII}^* = ((Y_{wik}^*)) = \left(Y_{w(n+1)k}^* \quad Y_{w(n+2)k}^* \quad \cdots \quad Y_{wNk}^*\right)'$$

$$\mathbf{Z}_S = \begin{pmatrix} Z_{01} & Z_{11} & \cdots & Z_{p1} \\ Z_{02} & Z_{12} & \cdots & Z_{p2} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{0n} & Z_{1n} & \cdots & Z_{pn} \end{pmatrix} \qquad \text{where } \mathbf{Z}_S = \begin{pmatrix} \mathbf{I}_n & \underset{n\times N-n}{\mathbf{0}} \end{pmatrix}\mathbf{Z}$$

$$\mathbf{Z}_R = \begin{pmatrix} Z_{0,n+1} & Z_{1,n+1} & \cdots & Z_{p,n+1} \\ Z_{0,n+2} & Z_{1,n+2} & \cdots & Z_{p,n+2} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{0,N} & Z_{1,N} & \cdots & Z_{p,N} \end{pmatrix} \qquad \text{where } \mathbf{Z}_R = \begin{pmatrix} \underset{(N-n)xn}{\mathbf{0}} & \mathbf{I}_{N-n} \end{pmatrix}\mathbf{Z}$$

$\mathbf{Z}_I = vec(\mathbf{Z}_S)$ Sample (Wenjun)

$\mathbf{Z}_{II} = vec(\mathbf{Z}_R)$ Remainder (Wenjun)

$\mathbf{Z}_I^+ = \mathbf{K}_I \mathbf{R}\mathbf{Z}^+$ Sample random variables in regression (Luz Mery)

$\mathbf{Z}_{II}^+ = \mathbf{K}_{II} \mathbf{R}\mathbf{Z}^+$ Remainder random variables in regression (Luz Mery)

$\tilde{\mathbf{Z}}_I = \left(\tilde{Y}_1 \quad \tilde{Y}_2 \quad \cdots \quad \tilde{Y}_n \;\vdots\; (X_1 - \mu_x) \quad (X_2 - \mu_x) \quad \cdots \quad (X_n - \mu_x) \right)'$ Regression with position error (Luz Mery)

<u>Expected Values, Targets, Predictors</u>

$P = \mathbf{g}' \begin{pmatrix} \mathbf{Y}_I \\ \mathbf{Y}_{II} \end{pmatrix}$ Target in problems with a simple population

$\theta = \mathbf{c}'\mathbf{Y} = \sum_{i=1}^N c_i Y_i$ Target in problems with auxiliary variables, by Wenjun.

$\theta = \mathbf{G}'\mathbf{Y}$ Target in domain problems, by Viviana

$\theta = \mathbf{G}_*'\mathbf{y}$ Alternative target in domain problems, by Viviana

$T = \mathbf{c}'\mathbf{Y} = \sum_{i=1}^N c_i Y_i$ Target in regression problem with error by Luz Mery

$\hat{P} = (\mathbf{g}_I' + \mathbf{a}')\mathbf{Y}_I$ Best linear unbiased predictor of target

$\hat{P} = \mathbf{w}'\mathbf{Z}_I$ Best linear unbiased predictor of target with auxiliary variables (Wenjun)

$\hat{T}^+ = (\mathbf{g}_I' + \hat{\mathbf{a}}')\mathbf{Z}_I^+$ Regression predictor (Luz Mery)

$\mathbf{a}'$ Constants to determine to optimally predict the target

$f(\mathbf{a}, \boldsymbol{\lambda}) = \mathbf{a}'\mathbf{V}_I\mathbf{a} - 2\mathbf{g}_{II}'\mathbf{V}_{II,I}\mathbf{a} + \mathbf{g}_{II}'\mathbf{V}_{II}\mathbf{g}_{II} + 2(\mathbf{a}'\mathbf{X}_I - \mathbf{g}_{II}'\mathbf{X}_{II})\boldsymbol{\lambda}$ Function to minimize for EMSE

$\boldsymbol{\lambda}$ LaGrangian multipliers

$$\hat{\mathbf{w}}' = \mathbf{V}_I^{-1}\left\{\mathbf{V}_{I,II} + \mathbf{G}_I\left(\mathbf{G}_I'\mathbf{V}_I^{-1}\mathbf{G}_I\right)^{-1}\left(\mathbf{G}_{II}' - \mathbf{G}_I'\mathbf{V}_I^{-1}\mathbf{V}_{I,II}\right)\right\}\mathbf{C}_{II}$$

$$= N\bar{c}_{II}\left(\left(1-f\right)\frac{\mathbf{1}_n'}{n} \;\vdots\; -\boldsymbol{\beta}'\otimes\frac{\mathbf{1}_n'}{n}\right)$$

Min EMSE constants with auxiliary

$$\hat{\mathbf{a}} = \left[\mathbf{V}_I^{-1} - \mathbf{V}_I^{-1}\mathbf{X}_I\left(\mathbf{X}_I'\mathbf{V}_I^{-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{-1}\right]\mathbf{V}_{I,II}\mathbf{g}_{II} + \mathbf{V}_I^{-1}\mathbf{X}_I\left(\mathbf{X}_I'\mathbf{V}_I^{-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_{II}'\mathbf{g}_{II}$$ Min EMSE constants

$$\hat{\mathbf{a}} = \begin{pmatrix} 1 \\ -\frac{\sigma_{xy}}{\sigma_x^2}\end{pmatrix}\otimes\left[\frac{D^+}{n}\mathbf{1}_n - \left(1-k^+\right)\mathbf{C}_I\right] + \bar{c}_{II}\begin{pmatrix} 1 \\ -\frac{\sigma_{xy}}{\sigma_x^2}\end{pmatrix}\otimes\mathbf{1}_n$$ Min EMSE constants for regression with error (Luz Mery)

$$\tilde{\mathbf{a}} = \begin{pmatrix} 1 \\ -\frac{\sigma_{xy}}{\sigma_x^2}\end{pmatrix}\otimes\left[\frac{\tilde{D}}{n}\left(\bigoplus_{i=1}^{n}\tilde{k}_i^*\right)\mathbf{1}_n - \left[\bigoplus_{i=1}^{n}c_i\left(1-\tilde{k}_i\right)\right]\mathbf{1}_n\right] + \bar{c}_{II}\begin{pmatrix} 1 \\ -\frac{\sigma_{xy}}{\sigma_x^2}\end{pmatrix}\otimes\mathbf{1}_n$$ Regression (Luz Mery)

$$\hat{\boldsymbol{\theta}}^* = \mathbf{G}_R'\left(\mathbf{V}_R - \mathbf{V}_{RS}\mathbf{V}_S^{-1}\mathbf{V}_{SR}\right)\mathbf{G}_R + \mathbf{G}_R'\left(\mathbf{X}_R - \mathbf{V}_{RS}\mathbf{V}_S^{-1}\mathbf{X}_S\right)\left(\mathbf{V}_{RS}\mathbf{V}_S^{-1}\mathbf{V}_{SR}\right)^{-1}\left(\mathbf{X}_R - \mathbf{V}_{RS}\mathbf{V}_S^{-1}\mathbf{X}_S\right)\mathbf{G}_R$$ M-optimal

$\text{var}\left(\mathbf{k}'\hat{\boldsymbol{\theta}}^*\right) \le \text{var}\left(\mathbf{k}'\hat{\boldsymbol{\theta}}\right)$ M-optimality criteria for $\hat{\boldsymbol{\theta}}^*$, a linear, unbiased predictor

$$E_\xi\begin{pmatrix}\mathbf{Y}_I \\ \mathbf{Y}_{II}\end{pmatrix} = \begin{pmatrix}\mathbf{X}_I \\ \mathbf{X}_{II}\end{pmatrix}\boldsymbol{\alpha}$$

$$E\begin{pmatrix}\mathbf{Y}_S \\ \mathbf{Y}_R\end{pmatrix} = \begin{pmatrix}\mathbf{X}_S \\ \mathbf{X}_R\end{pmatrix}\boldsymbol{\beta}$$ Expected value used by Viviana for domain problems

$$E_{\xi R}\begin{pmatrix}\mathbf{Y}_I^* \\ \mathbf{Y}_{II}^*\end{pmatrix} = \begin{pmatrix}\mathbf{X}_I \\ \mathbf{X}_{II}\end{pmatrix}\boldsymbol{\alpha}$$

$$\text{var}_\xi\begin{pmatrix}\mathbf{Y}_I \\ \mathbf{Y}_{II}\end{pmatrix} = \begin{pmatrix}\mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II}\end{pmatrix}$$

$$\text{var}\begin{pmatrix}\mathbf{Y}_S \\ \mathbf{Y}_R\end{pmatrix} = \begin{pmatrix}\mathbf{V}_S & \mathbf{V}_{SR} \\ \mathbf{V}_{RS} & \mathbf{V}_R\end{pmatrix}$$ Variance used by Viviana for domain problems.

$$\text{var}_{\xi R}\begin{pmatrix}\mathbf{Y}_I^* \\ \mathbf{Y}_{II}^*\end{pmatrix} = \begin{pmatrix}\mathbf{V}_I^* & \mathbf{V}_{I,II}^* \\ \mathbf{V}_{II,I}^* & \mathbf{V}_{II}^*\end{pmatrix}$$

$$\text{var}_{\xi R}\begin{pmatrix}\mathbf{Y}_I^* \\ \hline \mathbf{Y}_I \\ \hline \mathbf{Y}_{II}\end{pmatrix} = \begin{pmatrix}\mathbf{V}_I^* & \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_I & \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II,I} & \mathbf{V}_{II}\end{pmatrix}$$

$$\mathbf{V}_{RI}^* = \mathbf{V}_I^* - \mathbf{V}_I$$

$$\mathbf{V}_I^* = \mathbf{I}_n\left(\sigma^2+\sigma_e^2\right) - \frac{1}{N}\mathbf{J}_n\sigma^2$$

$$\mathbf{V}_I^+ = \boldsymbol{\Sigma}\otimes\mathbf{P}_{n,N} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\bar{\sigma}_e^2\otimes\mathbf{I}_n$$ Regression partitioned variance (Luz Mery)

$$\mathbf{V}_{II}^+ = \boldsymbol{\Sigma}\otimes\mathbf{P}_{N-n,N} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\bar{\sigma}_e^2\otimes\mathbf{I}_{N-n}$$ Regression variance (Luz Mery)

$$\mathbf{V}_{I,II}^+ = \frac{-1}{N}\boldsymbol{\Sigma}\otimes\mathbf{1}_n\mathbf{1}'_{N-n}$$ Regression variance (Luz Mery)

Other Terms Defined for Predictors

$$\hat{\boldsymbol{\alpha}}^* = \left(\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{Y}_I^*$$

$$\hat{\boldsymbol{\alpha}} = \left(\mathbf{X}_I{}'\mathbf{V}_I^{-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I{}'\mathbf{V}_I^{-1}\mathbf{Y}_I$$

$$\hat{\alpha} = \left(\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{Y}_I^*$$

$$\tilde{\mu} = \sum_{i=1}^{n}\frac{\tilde{k}_i}{\left(\sum_{j=1}^{n}\tilde{k}_j\right)}\tilde{Y}_i$$ Weighted sample mean with measurement error associated with position

$$\tilde{\mu}^+ = \sum_{i=1}^{n}\frac{k_i^+}{\left(\sum_{j=1}^{n}k_j^+\right)}\tilde{Y}_i$$ Weighted sample mean for partially conditional RP model with measurement error

$$\mu_{Y_I^{\Diamond+}} = \sum_{i=1}^{n}\frac{k_i^\Diamond}{\left(\sum_{j=1}^{n}k_j^\Diamond\right)}Y_i^{\Diamond+}$$ Regression problem (Luz Mery)

$$\mu_{X_I} = \sum_{i=1}^{n}\frac{k_i^\Diamond}{\left(\sum_{j=1}^{n}k_j^\Diamond\right)}X_i$$ Regression problem (Luz Mery)

$$\hat{\mu}_y^+ = \bar{Y}_I^+ - \frac{\sigma_{xy}}{\sigma_x^2}\left(\bar{X}_I - \mu_x\right)$$ Mean used in MSE for Regression Estimator of Luz Mery

$$\hat{B}^+ = N\left[\bar{Y}_i^{\Diamond+} - \frac{\sigma_{y^\Diamond x}}{\sigma_x^2}\left(\bar{X}_I - \mu_x\right)\right]$$ Regression problem (Luz Mery)

$\tilde{B} = N\left[\mu_{Y_I^{\Diamond +}} - \frac{\sigma_{y^{\Diamond}x}}{\sigma_x^2}\left(\mu_{X_I} - \mu_x\right)\right]$ Regression problem (Luz Mery)

$k_i^{\Diamond} = \frac{\sigma_{y^{\Diamond}}^2\left(1-\rho_{xy^{\Diamond}}^2\right)}{\sigma_i^{\Diamond 2} + \sigma_{y^{\Diamond}}^2\left(1-\rho_{xy^{\Diamond}}^2\right)}$ Regression problem, (Luz Mery)

$\sigma_i^{\Diamond 2} = \sum_{s=1}^{N} U_{is} x_s^{\Diamond 2} \sigma_{se}^2$ Regression problem (Luz Mery)

Clustered/Stratified Population

Units and Response

$s = 1,\ldots,N$	Cluster Labels (used by Ed)
$t = 1,\ldots,M_s$	Unit label for cluster s (used by Ed)
$j = 1,\ldots,J$	Stratum labels (used by Viviana)
$\wp_j = \{1, 2, \cdots, N_j\}$	Unit labels for stratum j (used by Viviana)
$i = 1,\ldots,I$	Domain label in each stratum
N_{ij}	Number of units in domain i in stratum j
$N_j = \sum_{i=1}^{I} N_{ij}$	
$\square = \sum_{s=1}^{N} M_s$	Number of units in the population
$k = 1,\ldots,r_{st}$	Index for replication
y_{st} for $s = 1,\ldots,N$; $t = 1,\ldots,M_s$	Expected Response for unit t in cluster s
y_{jk} for $j = 1,\ldots,J$; $k = 1,\ldots,N_j$	Expected Response for unit k in stratum j
w_{sj}	Weight for the unit in position j in cluster s (Ed)
$w_s = w_{sj}$	Weight in special case that all position weights are equal in cluster s (Ed)
W_{stk} for $s = 1,\ldots,N$; $t = 1,\ldots,M_s$; $k = 1,\ldots,r_s$	Response error. for k^{th} replication of unit t in cluster s

$Y_{stk} = y_{st} + W_{stk}$ for $k = 1,...,r_s$ — Response for k^{th} replication of unit t in cluster s

<u>Vectors</u>

$\mathbf{y}_s = ((y_{st})) = \begin{pmatrix} y_{s1} & y_{s2} & \cdots & y_{sM_s} \end{pmatrix}'$ — Expected response for units in cluster s

$\mathbf{y}_j = ((y_{jk})) = \begin{pmatrix} y_{j1} & y_{j2} & \cdots & y_{jN_j} \end{pmatrix}'$ — Expected response for units in stratum j (Viviana)

$\mathbf{y} = ((\mathbf{y}'_s))' = \begin{pmatrix} \mathbf{y}'_1 & \mathbf{y}'_2 & \cdots & \mathbf{y}'_N \end{pmatrix}'$ — Expected response for all units and clusters.

$\boldsymbol{\varepsilon}_s = ((\varepsilon_{st})) = \begin{pmatrix} \varepsilon_{s1} & \varepsilon_{s2} & \cdots & \varepsilon_{sM_s} \end{pmatrix}'$ — Deviation of unit t from expected value of cluster s

$\boldsymbol{\varepsilon} = ((\boldsymbol{\varepsilon}'_s))' = \begin{pmatrix} \boldsymbol{\varepsilon}'_1 & \boldsymbol{\varepsilon}'_2 & \cdots & \boldsymbol{\varepsilon}'_N \end{pmatrix}'$ — Vector of deviations of units from cluster expected values

$\mathbf{W}_s = ((W_{stk})) = \begin{pmatrix} W_{s1k} & W_{s2k} & \cdots & W_{sMk} \end{pmatrix}'$ — Response error vector (for $r_{st} = 1$ for all $t = 1,...,M_s$) for cluster s

$\mathbf{W} = ((\mathbf{W}'_s))' = \begin{pmatrix} \mathbf{W}'_1 & \mathbf{W}'_2 & \cdots & \mathbf{W}'_N \end{pmatrix}'$ — Response error vector for all units and clusters (for $r_{st} = 1$ for all $s = 1,...,N$; $t = 1,...,M_s$).

Parameters and Constants

m_s — Sample size for cluster s (Ed)

n_j — Sample size for stratum j (Viviana)

$f_s = \dfrac{m_s}{M_s}$

$\mathbf{K}_I = \left(\mathbf{I}_n \;\vdots\; \underset{n\times(N-n)}{\mathbf{0}}\right) \otimes \left(\mathbf{I}_m \;\vdots\; \underset{m\times(M-m)}{\mathbf{0}}\right)$ — Matrix to form sample response vector when $M_s = M$ for all $s = 1,...,N$

$$\mathbf{K}_{II} = \begin{pmatrix} \left(\mathbf{I}_n \;\vdots\; \underset{n\times(N-n)}{\mathbf{0}}\right) \otimes \left(\underset{(M-m)\times m}{\mathbf{0}} \;\vdots\; \mathbf{I}_{M-m}\right) \\ \left(\underset{(N-n)\times n}{\mathbf{0}} \;\vdots\; \mathbf{I}_{N-n}\right) \otimes \mathbf{I}_M \end{pmatrix}$$

Matrix to form remainder response vector when $M_s = M$ for all $s = 1,...,N$

$$\mathbf{K} = \left(\mathbf{K}_I' \;\vdots\; \mathbf{K}_{II}'\right)'$$

$$\mathbf{X}_I = \mathbf{K}_I \mathbf{X}$$

$$\mathbf{X}_I = \left[\frac{1}{N}\mathbf{1}_n \otimes \left(\bigoplus_{s=1}^{N} w_s m_s\right)\right]$$ Used for sample in Partially collapsed unequal size model (Ed)

$$\mathbf{X}_{II} = \mathbf{K}_{II} \mathbf{X}$$

$$\mathbf{X}_{II} = \begin{pmatrix} \frac{1}{N}\mathbf{1}_{N-n} \otimes \left(\bigoplus_{s=1}^{N} w_s m_s\right) \\ \hdashline \frac{1}{N}\mathbf{1}_N \otimes \left(\bigoplus_{s=1}^{N} w_s \left(M_s - m_s\right)\right) \end{pmatrix}$$ Used for remainder in Partially collapsed unequal size model (Ed)

$$\mathbf{g}' = \mathbf{c}' \otimes \mathbf{1}'_{\square}$$ Two-stage unbalanced problem (Ed)

$$\mathbf{g}_I = \mathbf{K}_I \mathbf{g}$$

$$\mathbf{g}_I' = \mathbf{c}_I' \otimes \mathbf{1}_N'$$ Coefficients for partially expanded sample (Ed)

$$\mathbf{g}_{II} = \mathbf{K}_{II} \mathbf{g}$$

$$\mathbf{g}_{II}' = \left(\mathbf{c}_{II}' \otimes \mathbf{1}_N' \;\vdots\; \mathbf{c}' \otimes \mathbf{1}_N'\right)$$ Coefficients for partially expanded remainder (Ed)

$$\mathbf{g}_{II}' = \left(\mathbf{g}_{1,II}' \;\vdots\; \mathbf{g}_{2,II}'\right)$$

$$\mathbf{g}_{1,II}' = \mathbf{g}_{II}' \left[\begin{pmatrix} \mathbf{I}_n \\ \hdashline \underset{(N-n)\times n}{\mathbf{0}} \end{pmatrix} \otimes \begin{pmatrix} \underset{m\times(M-m)}{\mathbf{0}} \\ \hdashline \mathbf{I}_{M-m} \end{pmatrix}\right]$$

$$\mathbf{g}_{2,II}' = \mathbf{g}_{II}' \left[\begin{pmatrix} \underset{n\times(N-n)}{\mathbf{0}} \\ \hdashline \mathbf{I}_{N-n} \end{pmatrix} \otimes \mathbf{I}_M\right]$$

$\mathbf{G}_S'$ Constants to define target in stratified domain problem for sample (Viviana)

$\mathbf{G}_R'$ Constants to define target in stratified domain problem for remainder (Viviana)

$$\mathbf{e}_i' = \begin{pmatrix} \underset{1\times n}{\mathbf{e}_{iI}'} & \vdots & \underset{1\times(N-n)}{\mathbf{e}_{iII}'} \end{pmatrix}$$

$$\mathbf{g}_i = \left(\left(g_{ij}\right)\right) = \begin{pmatrix} g_{i1} & g_{i2} & \cdots & g_{iM} \end{pmatrix}'$$ Constant multiplier to define target when $M_s = M$ for all $s = 1, \ldots, N$

$\mathbf{g}=\left(\left(\mathbf{g}_i\right)\right)'=\left(\mathbf{g}_1' \quad \mathbf{g}_2' \quad \cdots \quad \mathbf{g}_N'\right)'$ Constant multiplier to define target

$\mathbf{b}=\left(\left(b_i\right)\right)=\left(b_1 \quad b_2 \quad \cdots \quad b_N\right)'$ Vector of constants to define target

$$\mathbf{b}'=\left(\begin{array}{c|c} \underset{1\times n}{\mathbf{b}_I'} & \underset{1\times(N-n)}{\mathbf{b}_{II}'} \end{array}\right)$$

$\mu_s=\frac{1}{M_s}\sum_{t=1}^{M_s} y_{st}$ Mean for cluster s (Ed)

$d_s=M_s w_s \mu_s$ Used in partially collapsed unequal cluster sizes (Ed)

$\beta_j=\frac{1}{N_j}\sum_{k=1}^{N_j} y_{jk}$ Mean for stratum j (Viviana)

$$\mu=\frac{1}{N}\sum_{s=1}^{N}\mu_s$$

$$\beta_s=\mu_s-\mu$$

$\varepsilon_{st}=y_{st}-\mu_s$ so that $y_{st}=\mu+\beta_s+\varepsilon_{st}$

$$\sigma^2=\frac{1}{N-1}\sum_{s=1}^{N}\left(\mu_s-\mu\right)^2$$

$\sigma_s^2=\frac{1}{M_s-1}\sum_{t=1}^{M_s}\left(y_{st}-\mu_s\right)^2$ Variance of units in cluster s (Ed)

$v_{se}^{*2}=\left(\frac{1-f_s}{f_s}\right)\frac{M_s w_s^2 \sigma_s^2}{N}$ Used in partially collapsed expanded unequal cluster (Ed)

$\sigma_j^2=\frac{1}{N_j-1}\sum_{k=1}^{N_j}\left(y_{jk}-\mu_j\right)^2$ Variance of units in stratum j (Viviana)

$\sigma_e^2=\frac{1}{N}\sum_{s=1}^{N}\sigma_s^2$ Average unit variance over clusters

$\bar{\sigma}_s^2=\frac{1}{M_s}\sum_{t=1}^{M_s}\sigma_{st}^2$ Average replication variance over units in cluster s

$$\sigma_r^2=\frac{1}{NM_s}\sum_{s=1}^{N}\sum_{t=1}^{M_s}\sigma_{st}^2$$

Other Variances for Other Mixed Model Predictors

$v_i = \sigma^2 + \frac{\sigma_i^2}{m_i}$	variance for mixed model predictors
$w_i = \frac{1/v_i}{\sum_{i^*=1}^{n} 1/v_{i^*}}$	Weight for variance in mixed model predictors
$k_i = \frac{\sigma^2}{v_i}$	Shrinkage constant for mixed model predictor
δ^2	Variance of cluster means for Scott and Smith predictor
$v_i^* = \delta^2 + \frac{\sigma_i^2}{m_i}$	Variance for Scott and Smith's predictor
$w_i^* = \frac{1/v_i^*}{\sum_{i=1}^{n} 1/v_i^*}$	Weight for variance in Scott and Smith's predictor
$k_i^* = \frac{m_i\delta^2}{m_i\delta^2 + \sigma_i^2}$	Shrinkage constant for Scott and Smith's predictor
$\sigma^{*2} = \sigma^2 - \frac{\sigma_e^2}{M}$	Variance parameter defined for random permutation model
$\rho_s = \frac{\sigma^2}{\sigma^2 + \sigma_e^2}$	Intra class correlation of clusters
$\rho_t = \frac{\sigma_e^2}{\sigma_e^2 + \sigma_r^2}$	Intra class correlation of units with replication error
$k^* = \frac{m\sigma^{*2}}{m\sigma^{*2} + \left(\sigma_e^2 + \sigma_r^2\right)}$	Random permutation shrinkage constant with replication error where $M_s = M$ and $m_s = m$ for all $s = 1,...,N$
$k = \frac{m\sigma^{*2}}{m\sigma^{*2} + \sigma_e^2}$	Random permutation shrinkage constant for 2-stage cluster sampling with no measurement error where $M_s = M$ and $m_s = m$ for all $s = 1,...,N$.

$k_r^* = \dfrac{m\sigma^{*2} + \sigma_e^2}{m\sigma^{*2} + \sigma_e^2 + \sigma_r^2}$ Additional random permutation shrinkage factor when where $M_s = M$ and $m_s = m$ for all $s = 1, \ldots, N$

$k_s = \dfrac{d_s^2}{d_s^2 + (N-1)v_{se}^{*2}}$ Partially collapsed unequal cluster constant (Ed)

$\bar{k} = \dfrac{1}{N}\sum_{s=1}^{N} k_s$ Average using partially collapsed unequal cluster constant (Ed)

$k_s^* = k_s - \dfrac{k_s}{d_s}\dfrac{1}{N}\sum_{s^*=1}^{N}\left(\dfrac{1-k_{s^*}}{1-\bar{k}}\right)d_{s^*}$ Constant for partially collapsed unequal clusters (Ed)

<u>Vectors</u>

$\boldsymbol{\mu} = ((\mu_s)) = (\mu_1 \quad \mu_2 \quad \cdots \quad \mu_N)'$ Cluster means (Ed)

$\boldsymbol{\beta}' = (\beta_1 \quad \beta_2 \quad \cdots \quad \beta_N)$ Vector of cluster effects (Ed)

$\boldsymbol{\beta}' = (\beta_1 \quad \beta_2 \quad \cdots \quad \beta_J)$ Vector of strata means (Viviana)

$\mathbf{X} = \mathbf{1}_{\square}$ Fixed parameter design matrix

$\underset{\square \times N}{\mathbf{Z}} = \bigoplus_{s=1}^{N} \mathbf{1}_{M_s}$ Random effect design matrix

$\mathbf{w}_s = ((w_{sj})) = (w_{s1} \quad w_{s2} \quad \cdots \quad w_{sM_s})'$ Weight for positions in cluster s (Ed)

$\vec{\mathbf{C}}' = \begin{pmatrix} \bigoplus_{i=1}^{N}\bigoplus_{s=1}^{N}\left(\mathbf{1}'_{m_s} \mid \mathbf{0}'_{M_s-m_s}\right) \\ \hline \bigoplus_{i=1}^{N}\bigoplus_{s=1}^{N}\left(\mathbf{0}'_{m_s} \mid \mathbf{1}'_{M_s-m_s}\right) \end{pmatrix}$ Partially Collapsing matrix (Ed)

$\mathbf{g}'_p = \mathbf{g}'\left[\vec{\mathbf{C}}\left(\vec{\mathbf{C}}'\vec{\mathbf{C}}\right)^{-1}\right]$ Linear constants for target to apply to partially collapsed random variables (Ed)

$\mathbf{P}_{\vec{\mathbf{C}}} = \mathbf{I}_{N^2\bar{M}} - \vec{\mathbf{C}}\left(\vec{\mathbf{C}}'\vec{\mathbf{C}}\right)^{-1}\vec{\mathbf{C}}'$ Ortho-complement of partial collapsing (Ed)

Permutation Random Variables

$i = 1, \ldots, N$ Position in permutation of clusters

$j = 1, \ldots, M_s$ Position in permutation of units in cluster s

$\upsilon = 1,\ldots,N_j$ Position in permutation of units in stratum j

U_{is} for $i = 1,\ldots,N$; $s = 1,\ldots,N$ Indicator of selection of cluster s in position i.

$U_{jt}^{(s)}$ for $j = 1,\ldots,M_s$, $t = 1,\ldots,M_s$ and $s = 1,\ldots,N$ Indicator of selection of unit t in position j in cluster s

$U_{\upsilon k}^{(j)}$ for $\upsilon = 1,\ldots,N_j$, $k = 1,\ldots,N_j$ and $j = 1,\ldots,J$ Indicator of selection of unit k in position υ in stratum j (Viviana)

$\tilde{Y}_{sj} = \sum_{t=1}^{M_s} U_{jt}^{(s)} y_{st}$ Expected response SSU j in cluster s

$Y_{j\upsilon} = \sum_{k=1}^{N_j} U_{\upsilon k}^{(j)} y_{jk}$ Response of unit in position υ of stratum j

$\tilde{Y}_{wsj} = w_{sj} \sum_{t=1}^{M_s} U_{jt}^{(s)} y_{st}$ or $\tilde{Y}_{wsj} = w_{sj} \mathbf{y}_s' \mathbf{U}_j^{(s)}$ Expected weighted response for SSU j in cluster s (Ed)

$$Y_{ij} = \sum_{s=1}^{N} \sum_{t=1}^{M} U_{is} U_{jt}^{(s)} y_{st}$$

$$Y_{ijk}^* = Y_{ij} + W_{ijk}^*$$

$$B_i = \sum_{s=1}^{N} U_{is} \beta_s$$

<u>Vectors and Matrices</u>

$$\mathbf{U}_t^{(s)} = \left(\left(U_{jt}^{(s)}\right)\right) = \begin{pmatrix} U_{1t}^{(s)} & U_{2t}^{(s)} & \cdots & U_{Mt}^{(s)} \end{pmatrix}'$$

$$\mathbf{U}_j^{(s)} = \left(\left(U_{jt}^{(s)}\right)\right) = \begin{pmatrix} U_{j1}^{(s)} & U_{j2}^{(s)} & \cdots & U_{jM}^{(s)} \end{pmatrix}'$$

$\mathbf{U}^{(s)} = \left(\left(\mathbf{U}_j'^{(s)}\right)\right) = \begin{pmatrix} \mathbf{U}_1^{(s)} & \mathbf{U}_2^{(s)} & \cdots & \mathbf{U}_M^{(s)} \end{pmatrix}'$ Permute units in cluster s (Ed)

$\mathbf{U}^{(j)} = \left(\left(U_{\upsilon k}^{(j)}\right)\right) = \begin{pmatrix} U_{11}^{(j)} & U_{12}^{(j)} & \cdots & U_{1N_j}^{(j)} \\ U_{21}^{(j)} & U_{22}^{(j)} & & U_{2N_j}^{(j)} \\ & & & \\ U_{N_j 1}^{(j)} & U_{N_j 2}^{(j)} & & U_{N_j N_j}^{(j)} \end{pmatrix}$ Permute units in stratum j (Viviana).

$\mathbf{U}_s = ((U_{is})) = (U_{1s} \quad U_{2s} \quad \cdots \quad U_{Ns})'$ Note that this is a column of $\mathbf{U}$, while $\mathbf{U}_i'$ is a row of $\mathbf{U}$

$\mathbf{U} = ((\mathbf{U}_s'))' = (\mathbf{U}_1 \quad \mathbf{U}_2 \quad \cdots \quad \mathbf{U}_N)$

$\mathbf{U} = (\mathbf{U}_I \quad \mathbf{U}_{II})$ Partitioned cluster permutation matrix (Ed)

$\mathbf{U}_I = ((\mathbf{U}_i)) = (\mathbf{U}_1 \quad \mathbf{U}_2 \quad \cdots \quad \mathbf{U}_n)$ Sample portion of cluster permutation matrix (Ed)

$\mathbf{U}_{II} = ((\mathbf{U}_i)) = (\mathbf{U}_{n+1} \quad \mathbf{U}_{n+2} \quad \cdots \quad \mathbf{U}_N)$ Remainder portion of cluster permutation matrix (Ed)

$\tilde{\mathbf{Y}}_s = ((\tilde{Y}_{sj})) = (\tilde{Y}_{s1} \quad \tilde{Y}_{s2} \quad \cdots \quad \tilde{Y}_{sM_s})'$ or $\tilde{\mathbf{Y}}_s = \mathbf{U}^{(s)}\mathbf{y}_s = ((\tilde{Y}_{sj}))$ Permuted units in cluster s (Ed)

$\mathbf{Y}_j = \mathbf{U}^{(j)}\mathbf{y}_j = ((Y_{j\upsilon}))$ Permuted units in stratum j (Viviana)

$\mathbf{Y} = ((\mathbf{Y}_j)) = \left(\mathbf{Y}_1' \quad \mathbf{Y}_2' \quad \cdots \quad \mathbf{Y}_J'\right)'$ Vector of strata with permuted units (Viviana)

$\tilde{\mathbf{Y}}_{ws} = ((\tilde{Y}_{wsj})) = (\tilde{Y}_{ws1} \quad \tilde{Y}_{ws2} \quad \cdots \quad \tilde{Y}_{wsM_s})'$

$\mathbf{Y}_i = ((Y_{ij})) = (Y_{i1} \quad Y_{i2} \quad \cdots \quad Y_{iM})'$ When $M_s = M$ for all $s = 1,\ldots,N$.

$\mathbf{Y} = ((\mathbf{Y}_i)) = \left(\mathbf{Y}_1' \quad \mathbf{Y}_2' \quad \cdots \quad \mathbf{Y}_N'\right)'$ When $M_s = M$ for all $s = 1,\ldots,N$.

or $\mathbf{Y} = ((Y_{ij})) = (\mathbf{U} \otimes \mathbf{I}_M)\left(\bigoplus_{s=1}^{N} \mathbf{U}^{(s)}\right)\mathbf{y}$

$\mathbf{E} = ((E_{ij})) = (\mathbf{U} \otimes \mathbf{I}_M)\left(\bigoplus_{s=1}^{N} \mathbf{U}^{(s)}\right)\boldsymbol{\varepsilon}$ When $M_s = M$ for all $s = 1,\ldots,N$

$\mathbf{W}^* = ((W_{ijk}^*)) = (\mathbf{U} \otimes \mathbf{I}_M)\left(\bigoplus_{s=1}^{N} \mathbf{U}^{(s)}\right)\mathbf{W}$ When $M_s = M$ for all $s = 1,\ldots,N$ and when $k = 1,\ldots,r_{st}$

and $r_{st} = 1$ for all $s = 1,\ldots,N$; $t = 1,\ldots,M_s$;..

$\mathbf{B} = ((B_i)) = (B_1 \quad B_2 \quad \cdots \quad B_N)'$ or $\mathbf{B} = \mathbf{U}\boldsymbol{\beta}$

$\mathbf{Y}^* = \mathbf{Y} + \mathbf{W}^*$

$T = \mathbf{g}'\mathbf{Y}$ Target

$T_A = \mathbf{g}'\mathbf{Y}^*$ Target with measurement error

$T = \mathbf{g}'\overleftrightarrow{\mathbf{Y}}_w$ Target (2-stage unequal) (Ed)

$T = \mathbf{g}_p' \vec{\mathbf{Y}}_{wp}$ Target (2-stage partially collapsed) (Ed)

Expanded Random Variables

Vectors

$\vec{\mathbf{Y}}_{wi} = \left(\left(U_{is}\tilde{\mathbf{Y}}_{ws}\right)\right) = \begin{pmatrix} U_{i1}\tilde{\mathbf{Y}}_{w1}' & U_{i2}\tilde{\mathbf{Y}}_{w2}' & \cdots & U_{iN}\tilde{\mathbf{Y}}_{wN}' \end{pmatrix}'$

$\vec{\mathbf{E}}_{wi} = \vec{\mathbf{Y}}_{wi} - E_{\xi_2|\xi_1}\left(\vec{\mathbf{Y}}_{wi}\right)$ (Ed)

$\vec{\mathbf{Y}}_{w} = \left(\left(\vec{\mathbf{Y}}_{wi}\right)\right) = \begin{pmatrix} \vec{\mathbf{Y}}_{w1}' & \vec{\mathbf{Y}}_{w2}' & \cdots & \vec{\mathbf{Y}}_{wN}' \end{pmatrix}'$ (Ed)

$\vec{\mathbf{E}}_{w} = \left(\left(\vec{\mathbf{E}}_{wi}\right)\right) = \begin{pmatrix} \vec{\mathbf{E}}_{w1}' & \vec{\mathbf{E}}_{w2}' & \cdots & \vec{\mathbf{E}}_{wN}' \end{pmatrix}'$

$\vec{\mathbf{Y}}_{wp} = \vec{\mathbf{C}}'\vec{\mathbf{Y}}_{w}$ Partially Collapsed Expanded random variables (Ed)

Sample and Remainder

n_j number of units in sample in stratum j (Viviana)

n_{ij} number of units in domain i in stratum j (Viviana)

m_i number of units in sample for the PSU in position i used in mixed models

M_i number of units in the PSU in position i used in mixed models

$f_i = \dfrac{m_i}{M_i}$ sampling fraction of SSUs in the PSU in position i used in mixed models

$f = \dfrac{m}{M}$ where $M_s = M$ and $m_s = m$ for all $s = 1,...,N$ (Ed)

$I_s = \sum_{i=1}^{n} U_{is}$ Indicator of inclusion of cluster s in the sample (Ed)

$\bar{Y}_{sI} = \dfrac{1}{m_s}\sum_{t=1}^{M_s} U_{jt}^{(s)} y_{st}$ Partially collapsed unequal clusters (Ed)

$\hat{Y}_i = \sum_{s=1}^{N} U_{is} M_s w_s k_s^* \bar{Y}_{sI}$ Weighted sample mean for PSU i (partially collapsed unequal) (Ed)

$\bar{\hat{Y}} = \dfrac{1}{n}\sum_{i=1}^{n}\hat{Y}_i$ Average for sample in partially collapsed unequal clusters (Ed)

$\bar{Y}^* = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ijk}^*$ where $M_s = M$ and $m_s = m$ for all $s = 1,\ldots,N$ and $k = 1,\ldots,r_{st}$

where $r_{st} = 1$ for all $s = 1,\ldots,N$; $t = 1,\ldots,M_s$

$\bar{Y} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij}$ where $M_s = M$ and $m_s = m$ for all $s = 1,\ldots,N$

$\bar{Y}_i^* = \frac{1}{m}\sum_{j=1}^{m} Y_{ijk}^*$ where $M_s = M$ and $m_s = m$ for all $s = 1,\ldots,N$ and $k = 1,\ldots,r_{st}$

where $r_{st} = 1$ for all $s = 1,\ldots,N$; $t = 1,\ldots,M_s$

$\bar{Y}_i = \frac{1}{m}\sum_{j=1}^{m} Y_{ij}$ where $M_s = M$ and $m_s = m$ for all $s = 1,\ldots,N$

Vectors

$$\mathbf{Y}_I = \mathbf{K}_I \mathbf{Y}$$

$$\mathbf{Y}_{II} = \mathbf{K}_{II} \mathbf{Y}$$

$$\mathbf{Y}_I^* = \mathbf{K}_I \mathbf{Y}^*$$

$$\mathbf{Y}_{II}^* = \mathbf{K}_{II} \mathbf{Y}^*$$

$$\bar{\mathbf{Y}}_I^* = \left(\left(\bar{Y}_i^*\right)\right) = \begin{pmatrix}\bar{Y}_1^* & \bar{Y}_2^* & \cdots & \bar{Y}_n^*\end{pmatrix}'$$

$$\bar{\mathbf{Y}}_I = \left(\left(\bar{Y}_i\right)\right) = \begin{pmatrix}\bar{Y}_1 & \bar{Y}_2 & \cdots & \bar{Y}_n\end{pmatrix}'$$

Expanded Random Variables

Vectors

$\overleftrightarrow{\mathbf{Y}}_{wI} = \begin{pmatrix}\mathbf{I}_{nN} & \mathbf{0}_{nN\times\left(2N^2-nN\right)}\end{pmatrix}\overleftrightarrow{\mathbf{Y}}_{wp}$ Sample random variables (Ed)

$\overleftrightarrow{\mathbf{Y}}_{wII} = \begin{pmatrix}\mathbf{0}_{\left(2N^2-nN\right)\times nN} & \mathbf{I}_{2N^2-nN}\end{pmatrix}\overleftrightarrow{\mathbf{Y}}_{wp}$ Remaining random variables (Ed)

$\overleftrightarrow{\mathbf{E}}_{wI} = \begin{pmatrix}\mathbf{I}_{nN} & \mathbf{0}_{nN\times\left(2N^2-nN\right)}\end{pmatrix}\overleftrightarrow{\mathbf{E}}_{w}$ Partitioned sample residuals (Ed)

$$\vec{\mathbf{E}}_{wII} = \begin{pmatrix} \mathbf{0}_{\left(2N^2-nN\right)\times nN} & \mathbf{I}_{2N^2-nN} \end{pmatrix} \vec{\mathbf{E}}_w$$ Partitioned remainder residuals (Ed)

$$\operatorname{var}_{\xi_1\xi_2}\begin{pmatrix} \vec{\mathbf{Y}}_{wI} \\ \vec{\mathbf{Y}}_{wII} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}'_{I,II} & \mathbf{V}_{II} \end{pmatrix}$$ Partially collapsed unequal cluster size (Ed)

$$\mathbf{V}_I = \frac{1}{N-1}\left(\mathbf{I}_n - \frac{1}{N}\mathbf{J}_n\right) \otimes \left[\left(\bigoplus_{s=1}^{N} f_s d_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N} f_s d_s\right)\right] + \mathbf{I}_n \otimes \left(\bigoplus_{s=1}^{N} f_s^2 v_{se}^{*2}\right) \quad \text{(Ed)}$$

$$\mathbf{V}_{I,II} = \left(\begin{array}{c:c} \frac{1}{N-1}\left(-\frac{1}{N}\mathbf{J}_{n\times(N-n)}\right)\otimes & \frac{1}{N-1}\left[\begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{n\times(N-n)} \end{pmatrix} - \frac{1}{N}\mathbf{J}_{n\times N}\right] \otimes \left[\left(\bigoplus_{s=1}^{N} f_s d_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N}(1-f_s)d_s\right)\right] \\ \left[\left(\bigoplus_{s=1}^{N} f_s d_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N} f_s d_s\right)\right] & -\begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{n\times(N-n)} \end{pmatrix}\otimes\left(\bigoplus_{s=1}^{N} f_s^2 v_{se}^{*2}\right) \end{array}\right) \quad \text{(Ed)}$$

$$\mathbf{V}_{II} = \frac{1}{N-1}\left(\begin{array}{c:c} \begin{gathered}\left(\mathbf{I}_{N-n} - \frac{1}{N}\mathbf{J}_{N-n}\right)\otimes \\ \left[\left(\bigoplus_{s=1}^{N} f_s d_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N} f_s d_s\right)\right]\end{gathered} & \begin{gathered}\left[\begin{pmatrix} \mathbf{0}_{(N-n)\times n} & \mathbf{I}_{N-n} \end{pmatrix} - \frac{1}{N}\mathbf{J}_{(N-n)\times N}\right]\otimes \\ \left[\left(\bigoplus_{s=1}^{N} f_s d_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N}(1-f_s)d_s\right)\right]\end{gathered} \\ \hdashline \begin{gathered}\left(\begin{pmatrix} \mathbf{0}_{n\times(N-n)} \\ \mathbf{I}_{N-n} \end{pmatrix} - \frac{1}{N}\mathbf{J}_{N\times(N-n)}\right)\otimes \\ \left[\left(\bigoplus_{s=1}^{N}(1-f_s)d_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N} f_s d_s\right)\right]\end{gathered} & \begin{gathered}\mathbf{P}_N \otimes \\ \left[\left(\bigoplus_{s=1}^{N}(1-f_s)d_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N}(1-f_s)d_s\right)\right]\end{gathered} \end{array}\right) + \left(\begin{array}{c:c:c} \mathbf{I}_{N-n} & \mathbf{0}_{(N-n)\times n} & -\mathbf{I}_{N-n} \\ \hdashline \begin{gathered}\mathbf{0}_{n\times(N-n)} \\ \hdashline -\mathbf{I}_{N-n}\end{gathered} & \multicolumn{2}{c}{\mathbf{I}_N} \end{array}\right)\otimes\left(\bigoplus_{s=1}^{N} f_s^2 v_{se}^{*2}\right)$$

(Ed)

Expected Values

$$E_{\xi_1\xi_2\xi_3}\left(\mathbf{Y}^*\right) = \mathbf{X}\mu$$

$\mathbf{X}_S$ Sample portion of design matrix for stratified domains (Viviana)

$\mathbf{X}_R$ Remainder portion of design matrix for stratified domains (Viviana)

$$\operatorname{var}_{\xi_1\xi_2}\begin{pmatrix} \mathbf{Y}_I \\ \hline \mathbf{Y}_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II} \end{pmatrix}$$

$$\operatorname{var}_{\xi_1\xi_2\xi_3}\begin{pmatrix} \mathbf{Y}_I^* \\ \hline \mathbf{Y}_I \\ \hline \mathbf{Y}_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_I^* & \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_I & \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II,I} & \mathbf{V}_{II} \end{pmatrix}$$

$$\mathbf{V}_I^* = \mathbf{V}_I + \sigma_r^2 \mathbf{I}_{nm}$$

Estimators/Predictors

$\bar{Y}_i = \frac{1}{m_i}\sum_{j=1}^{m_i} Y_{ij}$ indicates the average of SSUs for the PSU realized in position i in a sample. This notation is used with mixed models.

$\hat{\mu} = \sum_{i=1}^{n} w_i \bar{Y}_i$ weighted mean used in mixed models.

$\hat{T} = \mathbf{L}'\vec{\mathbf{Y}}_{wI}$ Predictor for partially collapsed unequal cluster (Ed)

$$\hat{\mathbf{L}} = \mathbf{g}_I + \left[\mathbf{V}_I^{-1} - \mathbf{V}_I^{-1}\mathbf{X}_I\left(\mathbf{X}_I{}'\mathbf{V}_I^{-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I{}'\mathbf{V}_I^{-1}\right]\mathbf{V}_{I,II}\mathbf{g}_{II} + \mathbf{V}_I^{-1}\mathbf{X}_I\left(\mathbf{X}_I{}'\mathbf{V}_I^{-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_{II}'\mathbf{g}_{II}$$

$$= \left(\mathbf{P}_n\mathbf{c}_I \otimes \left[\bigoplus_{s=1}^{N}\frac{k_s^*}{f_s}\right]\mathbf{1}_N\right) + \frac{N}{n}\bar{c}\left[\mathbf{1}_n \otimes \left(\bigoplus_{s=1}^{N}\frac{1}{f_s}\right)\mathbf{1}_N\right]$$

BLUP (Ed)

$\hat{T} = \sum_{i=1}^{n} c_i\hat{Y}_i - n\bar{c}_I\bar{\hat{Y}} + \frac{N}{n}\bar{c}\left[\sum_{s=1}^{N} I_s\left(M_s w_s \bar{Y}_{sI}\right)\right]$ Partially collapsed unequal cluster predictor (Ed)

$$\hat{\boldsymbol{\alpha}}^* = \left(\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{Y}_I^*$$

$$\hat{\sigma}_e^2 = \max\left(0\,,\, MSE - \sigma_r^2\right)$$

$$\hat{\sigma}^2 = \max\left(0\,,\frac{1}{m}\left[MSB - MSE + f\,\hat{\sigma}_e^2\right]\right)$$

$$\hat{\sigma}^{*2} = \max\left(0\,,\frac{1}{m}\left[MSB - MSE\right]\right)$$