Measure for measure: exact F tests and the mixed models controversy

Viviana B. Lencina
Departamento de Investigación, FM
Universidad Nacional de Tucumán, Argentina

Julio M. Singer
Departamento de Estatística, IME
Universidade de São Paulo, Brazil

Abstract

We consider exact F tests for the hypothesis of null random factor effect in the presence of interaction under the two factor mixed models involved in the mixed models controversy. We show that under the constrained parameter (CP) model, even in unbalanced data situations, MSB/MSE (in the usual $ANOVA$ notation) follows an exact F distribution when the null hypothesis holds. We also obtain an exact F test for what is generally (and erroneously) assumed to be an equivalent hypothesis under the unconstrained parameter (UP) model. For unbalanced data, such the corresponding test statistic does not coincide with $MSB/MSAB$ (the test statistic advocated for balanced data cases). We compute the power of the exact test under different imbalance patterns and show that although the loss of power increases with the degree of imbalance, it still remains reasonable from a practical point of view.

Key words: Exact test, Householder orthogonal transformation, mixed model, random effect, variance components.

1 Introduction

Mixed models are mainly considered for the analysis of observational studies, although they may also be employed under some experimental designs as indicated in Hinkelmann and Kempthorne (1994, Sec. 9.7.5), for example. The so-called mixed model controversy arises in situations where the data can be described by means of a two-way classification with a fixed factor A, with a levels, a random factor B, with b levels and r replicates. In this context, two mixed models have been commonly employed. The *unconstrained parameters* (UP) *model* is

$$Y_{ijk} = \mu + \alpha_i + B_j + (\alpha B)_{ij} + E_{ijk}, \qquad (1)$$

for $i = 1, ..., a$, $j = 1, .., b$ and $k = 1, ..., r$, with $(B_1, ..., B_b)^t \sim N(\mathbf{0}, \sigma^2_B \mathbf{I}_b)$, $((\alpha B)_{11}, ..., (\alpha B)_{ab})^t \sim N(\mathbf{0}, \sigma^2_{\alpha B} \mathbf{I}_{ab})$, $(E_{111}, ..., E_{1br}, , ..., E_{a11}, ..., E_{abr})^t \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{abr})$ representing independent random vectors and $\mathbf{I}_n$ denoting an $n \times n$ identity matrix. The second model is

$$Y_{ijk} = \eta + \tau_i + D_j + (\tau D)_{ij} + E_{ijk}, \qquad (2)$$

for $i = 1, ..., a$, $j = 1, .., b$ and $k = 1, ..., r$, with $\sum_{i=1}^{a} \tau_i = 0$, $(D_1, ..., D_b)^t \sim N(\mathbf{0}, \sigma^2_D \mathbf{I}_b)$, $((\tau D)_{11}, ..., (\tau D)_{ab})^t \sim N(\mathbf{0}, \sigma^2_{\tau D} \mathbf{P}_a \otimes \mathbf{I}_b)$ and $(E_{111}, ..., E_{1br}, ..., E_{abr})^t \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{abr})$ representing independent random vectors, where $\otimes$ denotes the Kronecker product, $\mathbf{P}_n = \mathbf{I}_n - n^{-1}\mathbf{J}_n$ and $\mathbf{J}_n$ denotes an $n \times n$ matrix with all elements equal to 1. Under model (2), $Cov((\tau D)_{ij}, (\tau D)_{i'j}) = -\sigma^2_{\tau D}/a$ for $i \neq i'$. This setup may be reproduced by assuming that the D_i, $(\tau D)_{ij}$ and E_{ijk} are independent of each other, that $E_{ijk} \sim N(\mathbf{0}, \sigma^2)$ are independent, that $D_i \sim N(\mathbf{0}, \sigma^2_D)$, $i = 1, ..., a$, are independent, and that $(\tau D)_{ij} \sim N(\mathbf{0}, \sigma^2_{\tau D})$, $i = 1, ..., a$, $j = 1, .., b$ with $\sum_{i=1}^{a} (\tau D)_{ij} = 0$, $j = 1, .., b$. Because of the restriction on the $(\tau D)_{ij}$, the model is termed *constrained parameters* (*CP*) *model*. Both models have been considered by many authors, like Searle (1971), Milliken and Johnson (1984), SAS (1990), Montgomery (1997), Neter et al. (1996), Voss (1999) and Lencina et al. (2005).

The controversy relates to the different statistics obtained under the two models for testing what presumably is the hypothesis of nonexistence of the random factor main effects in the presence of interaction. The major problem is that one may obtain conflicting results. More specifically, under the *UP* model, $H_0 : \sigma^2_B = 0$ is interpreted as the hypothesis of nonexistence of the random factor main effects and $MSB/MSAB$ (in the usual *ANOVA* notation) is the appropriate test statistic; under the *CP* model, $H_0 : \sigma^2_D = 0$ is interpreted similarly and the standard test rejects H_0 if MSB/MSE is sufficiently large. Voss (1999) and Lencina et al. (2005), using different arguments, conclude that only MSB/MSE is suitable to test the hypothesis of interest, i.e., of the nonexistence of the random factor main effects in the presence of interaction. In fact, Hocking (1973) and Lencina et al. (2005) argue that the problem is not to choose what model to use but how to express the hypothesis in terms of the model parameters. The latter do not argue against the *UP* model but do argue against using $H_0 : \sigma_B^2 = 0$ as the hypothesis of nonexistence of the random factor main effects in the presence of the interaction.

Many authors, like Hinkelmann (2000) and Wolfinger and Stroup (2000) mention that the *UP* model is being used more extensively because of the availability of commercial software for analysis using it. In particular, they argue that it may be employed for unbalanced data, i.e. (1) and (2) with $k = 1, ..., n_{ij}$, where $n_{ij} > 0$, $i = 1, ..., a$, $j = 1, ..., b$ is the number of observations under level i of factor A and level j of factor B and recommend its use along with likelihood-based inferential procedures. The picture for unbalanced data, however, is not as clear as suggested by these authors. Although it is true that *UP* model based software, like Proc Mixed in SAS, are useful tools for the analysis of unbalanced mixed models, we must keep in mind that usually they do not provide exact tests for null variance components under such conditions. The available Wald and likelihood ratio tests are only approximations, since they rely on asypmtotic arguments. Moreover, since under the null hypothesis, the parameter lies in the boundary of the parametric space, such large sample tests do not follow chi-squared distributions as indicated in Stram and Lee (1994) and more recently in Crainiceanu and Ruppert (2004). The question addressed in this paper is whether we can obtain exact tests for the relevant hypotheses under the models involved in the controversy.

In Section 2 we introduce the general linear mixed model and specify it for the *UP* and the *CP* models; then we show that under the *CP* model, MSB/MSE obtained from the $ANOVA$ table has an exact F distribution when the hypothesis $H_0 : \sigma_D^2 = 0$ holds, even in unbalanced data cases. In Section 3 we consider the transformation proposed by Öfversten (1993) to obtain exact F tests for the hypotheses $H_0 : \sigma_B^2 = 0$, under the *UP* model and $H_0 : \sigma_D^2 = 0$, under the *CP* model. This is in line with Lencina et al. (2005), who observe that the test statistics coincide with those of the $ANOVA$ table either for balanced or unbalanced data under the *CP* model, but not under the *UP* model. In Section 4, we consider a simulation study to evaluate the power of the test for $H_0 : \sigma_D^2 = 0$ under different unbalanced data patterns and compare the simulated power with that obtained under balanced situations; we also obtain a lower bound to the power of the tests. We summarize the results in Section 5.

2 The general linear mixed model specification

The general linear mixed model may be written as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{b} + \mathbf{E}, \tag{3}$$

where $\mathbf{Y}$ is the observed vector of responses, $\boldsymbol{\beta}$ is a vector of unknown fixed effects, $\mathbf{b}$ is a vector of unobservable random effects, $\mathbf{X}$ is the model specification matrix for the fixed effects, $\mathbf{Z}$ is the model specification matrix for the random effects, and $\mathbf{E}$ is a vector of unobservable random errors.

For unbalanced data, the *UP* model may be expressed as (3) with $\mathbf{Y} = (Y_{111}, ..., Y_{1bn_{1b}}, ..., Y_{a11}, ..., Y_{abn_{ab}})^t$, $\boldsymbol{\beta} = (\mu, \alpha_1, ..., \alpha_a)^t$, $\mathbf{b} = (\mathbf{b}_1^t \quad \mathbf{b}_2^t)^t$ where $\mathbf{b}_1 = (B_1, ..., B_b)^t \sim N(\mathbf{0}, \sigma_B^2 \mathbf{I}_b)$, $\mathbf{b}_2 = ((\alpha B)_{11}, ..., (\alpha B)_{1b}, ..., (\alpha B)_{a1}, ..., (\alpha B)_{ab})^t \sim N(\mathbf{0}, \sigma_{\alpha B}^2 \mathbf{I}_{ab})$ and $\mathbf{E} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{n_{..}})$ are independent random vectors, $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]$, with $\mathbf{X}_1 = \mathbf{1}_{n_{..}}$, $\mathbf{X}_2 = \bigoplus_{i=1}^{a} \mathbf{1}_{n_{i.}}$, where $\bigoplus$ denotes the direct sum (Searle, 1982), and $\mathbf{Z} = [\mathbf{Z}_1,\ \mathbf{Z}_2]$, with $\mathbf{Z}_1^t = [\mathbf{Z}_{11}^t,\ \mathbf{Z}_{12}^t, \ldots, \mathbf{Z}_{1a}^t]$ where $\mathbf{Z}_{1i} = \bigoplus_{j=1}^{b} \mathbf{1}_{n_{ij}}$ for $i = 1, ..., a$, and $\mathbf{Z}_2 = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b} \mathbf{1}_{n_{ij}}$. Here $\mathbf{1}_{n_{..}}$ is a column vector with all elements equal to 1, $n_{..} = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}$ is the total number of observations and $n_{i.} = \sum_{j=1}^{b} n_{ij}$ is the total number of observations under level i of factor A.

The *CP* model for unbalanced data may also be written as (3) with $\boldsymbol{\beta} = (\eta, \tau_1, \ldots, \tau_{a-1})^t$, $\mathbf{b} = [\mathbf{b}_1^t,\ \mathbf{b}_2^t]^t$, where $\mathbf{b}_1 = (D_1, \ldots, D_b)^t \sim N(\mathbf{0}, \sigma_D^2 \mathbf{I}_b)$, $\mathbf{b}_2 = ((\tau D)_{11}, \ldots, (\tau D)_{1b}, \ldots, (\tau D)_{(a-1)1}, \ldots, (\tau D)_{(a-1)b})^t \sim N(\mathbf{0}, \sigma_{\tau D}^2 (\mathbf{I}_{a-1} - a^{-1}\mathbf{J}_{a-1}) \otimes \mathbf{I}_b)$ and $\mathbf{E} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{n_{..}})$ are independent random vectors, $\mathbf{X} = [\mathbf{X}_1,\ \mathbf{X}_2]$, with $\mathbf{X}_1 = \mathbf{1}_{n_{..}}$, $\mathbf{X}_2^t = [\mathbf{X}_{21}^t,\ \mathbf{X}_{22}^t]$, $\mathbf{X}_{21} = \bigoplus_{i=1}^{a-1} \mathbf{1}_{n_{i.}}$, $\mathbf{X}_{22} = \mathbf{1}_{a-1}^t \otimes -\mathbf{1}_{n_{a.}}$, $\mathbf{Z} = [\mathbf{Z}_1,\ \mathbf{Z}_2]$, with $\mathbf{Z}_1^t = [\mathbf{Z}_{11}^t,\ \mathbf{Z}_{12}^t, \ldots, \mathbf{Z}_{1a}^t]$, $\mathbf{Z}_{1i} = \bigoplus_{j=1}^{b} \mathbf{1}_{n_{ij}}$, $\mathbf{Z}_2^t = [\mathbf{Z}_{21}^t,\ \mathbf{Z}_{22}^t]$, $\mathbf{Z}_{21} = \bigoplus_{i=1}^{a-1} \bigoplus_{j=1}^{b} \mathbf{1}_{n_{ij}}$ and $\mathbf{Z}_{22} = \mathbf{1}_{a-1}^t \otimes (\bigoplus_{j=1}^{b} -\mathbf{1}_{n_{aj}})$.

In both cases it follows that $\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \mathbf{Z}\mathbf{V}\mathbf{Z}^t + \sigma^2 \mathbf{I}_{n_{..}})$ where $\mathbf{V}$ is a block diagonal matrix. More specifically, for the *UP* model,

$$\mathbf{V} = \begin{bmatrix} \sigma_B^2 \mathbf{I}_b & \mathbf{0} \\ \mathbf{0} & \sigma_{\alpha B}^2 \mathbf{I}_{ab} \end{bmatrix} \quad \text{and} \quad \mathbf{Z}\mathbf{V}\mathbf{Z}^t = \sigma_B^2 \mathbf{Z}_1 \mathbf{Z}_1^t + \sigma_{\alpha B}^2 \mathbf{Z}_2 \mathbf{Z}_2^t$$

whereas, for the CP model

$$\mathbf{V} = \left[\begin{array}{cc} \sigma_D^2 \mathbf{I}_b & \mathbf{0} \\ \mathbf{0} & \sigma_{\tau D}^2 (\mathbf{I}_{a-1} - a^{-1}\mathbf{J}_{a-1}) \otimes \mathbf{I}_b \end{array} \right]$$

and

$$\mathbf{ZVZ}^t = \sigma_D^2 \mathbf{Z}_1 \mathbf{Z}_1^t + \sigma_{\tau D}^2 \mathbf{Z}_2 (\mathbf{I}_{a-1} - a^{-1}\mathbf{J}_{a-1}) \otimes \mathbf{I}_b \mathbf{Z}_2^t.$$

Using the notation of Kshirsagar (1983) and assuming that $n_{ij} > 0$, the type III sum of squares due to the random factor B is

$$SSB = \sum_{j=1}^{b} W_j^{'} (L_j^{'} - \bar{L}^{'})^2,$$

with $W_j^{'-1} = \sum_{i=1}^{a} 1/n_{ij}$, $L_j^{'} = \sum_{i=1}^{a} Y_{ij.}/n_{ij}$, $Y_{ij.} = \sum_{k=1}^{n_{ij}} Y_{ijk}$ and $\bar{L}^{'} = \sum_{j=1}^{b} W_j^{'} L_j^{'} / \sum_{j^*=1}^{b} W_{j^*}^{'}$. In matrix notation, it can be expressed as the quadratic form $SSB = \mathbf{Y}^t \mathbf{M}_B \mathbf{Y}$, where $\mathbf{M}_B = \mathbf{T}_1 \mathbf{A}^{-1} \mathbf{T}_1^t$ with $\mathbf{T}_1^t = (\mathbf{T}_{11}^t | \cdots | \mathbf{T}_{1a}^t)$,

$$\mathbf{T}_{1i} = \left[\begin{array}{c} \mathbf{1}_{(b-1)}^t \otimes n_{i1}^{-1} \mathbf{1}_{n_{i1}} \\ \oplus_{j=2}^{b} - n_{ij}^{-1} \mathbf{1}_{n_{ij}} \end{array} \right],$$

and $\mathbf{A} = \sum_{i=1}^{a} n_{i1}^{-1} \mathbf{J}_{b-1} + \bigoplus_{j=2}^{b} (\sum_{i=1}^{a} n_{ij}^{-1})$.

On the other hand, the error sum of squares may be expressed as $SSE = \mathbf{Y}^t \mathbf{M}_E \mathbf{Y}$, with $\mathbf{M}_E = \mathbf{I}_{n..} - \mathbf{T}_2 \mathbf{G} \mathbf{T}_2^t$, $\mathbf{G} = (\mathbf{T}_2^t \mathbf{T}_2)^-$, and

$$\mathbf{T}_2 = \left[\begin{array}{cccccccccc} \mathbf{1}_{n_{1.}} & \mathbf{1}_{n_{1.}} & \mathbf{0} & \cdots & \mathbf{0} & \bigoplus_{j=1}^{b} \mathbf{1}_{n_{1j}} & \bigoplus_{j=1}^{b} \mathbf{1}_{n_{1j}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1}_{n_{2.}} & \mathbf{0} & \mathbf{1}_{n_{2.}} & \cdots & \mathbf{0} & \bigoplus_{j=1}^{b} \mathbf{1}_{n_{2j}} & \mathbf{0} & \bigoplus_{j=1}^{b} \mathbf{1}_{n_{2j}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{n_{a.}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{n_{a.}} & \bigoplus_{j=1}^{b} \mathbf{1}_{n_{aj}} & \mathbf{0} & \mathbf{0} & \cdots & \bigoplus_{j=1}^{b} \mathbf{1}_{n_{aj}} \end{array} \right].$$

Also, under the null hypothesis, $H_0 : \sigma_D^2 = 0$, it follows that the covariance matrix of $\mathbf{Y}$ reduces to $\mathbf{V} = \sigma_{\tau_D}^2 \mathbf{V}_1 + \sigma^2 \mathbf{V}_2$, where

$$\mathbf{V}_1 = a^{-1} \begin{bmatrix} (a-1)\bigoplus_{j=1}^{b}\mathbf{J}_{n_{1j}} & -\bigoplus_{j=1}^{b}(\mathbf{1}_{n_{1j}}\mathbf{1}_{n_{2j}}^t) & \cdots & -\bigoplus_{j=1}^{b}(\mathbf{1}_{n_{1j}}\mathbf{1}_{n_{aj}}^t) \\ -\bigoplus_{j=1}^{b}(\mathbf{1}_{n_{2j}}\mathbf{1}_{n_{1j}}^t) & (a-1)\bigoplus_{j=1}^{b}\mathbf{J}_{n_{2j}} & \cdots & -\bigoplus_{j=1}^{b}(\mathbf{1}_{n_{2j}}\mathbf{1}_{n_{aj}}^t) \\ \vdots & \vdots & \ddots & \vdots \\ -\bigoplus_{j=1}^{b}(\mathbf{1}_{n_{aj}}\mathbf{1}_{n_{1j}}^t) & -\bigoplus_{j=1}^{b}(\mathbf{1}_{n_{aj}}\mathbf{1}_{n_{2j}}^t) & \cdots & (a-1)\bigoplus_{j=1}^{b}\mathbf{J}_{n_{aj}} \end{bmatrix}$$

and $\mathbf{V}_2 = \mathbf{I}_{n..}$.

To prove that both quadratic forms have chi-squared distributions multiplied by constants, it is enough to show that $(\sigma^2)^{-1}\mathbf{M}_B\mathbf{V}(\mathbf{Y})$ and $(\sigma^2)^{-1}\mathbf{M}_E\mathbf{V}(\mathbf{Y})$ are idempotent. In this direction, observe that $\mathbf{T}_1^t\mathbf{V}_1 = \mathbf{0}$ and accordingly, $\mathbf{M}_B\mathbf{V}_1 = \mathbf{0}$, so that

$$(\sigma^2)^{-1}\mathbf{M}_B\mathbf{V}(\mathbf{Y}) = \mathbf{M}_B\mathbf{I}_{n..} = \mathbf{M}_B,$$

with $\mathbf{M}_B$ being idempotent because $\mathbf{T}_1^t\mathbf{T}_1 = \mathbf{A}$. On the other hand, as the columns of $\mathbf{V}_1$ are linear combinations of the columns of $\mathbf{T}_2$, it follows that $\mathbf{T}_2\mathbf{G}\mathbf{T}_2^t\mathbf{V}_1 = \mathbf{V}_1$ and therefore, $\mathbf{M}_E\mathbf{V}_1 = \mathbf{0}$. Then we may conclude that

$$(\sigma^2)^{-1}\mathbf{M}_E\mathbf{V}(\mathbf{Y}) = \mathbf{M}_E\mathbf{I}_{n..} = \mathbf{M}_E,$$

which is an idempotent matrix. Consequently, it follows that SSB/σ^2 and SSE/σ^2 have chi-squared distributions under the CP model, when $\sigma_D^2 = 0$.

In addition, noting that $\mathbf{M}_E\mathbf{V}(\mathbf{Y})\mathbf{M}_B = \sigma^2\mathbf{M}_E\mathbf{M}_B$ and that the columns of $\mathbf{T}_1$ are linear combinations of the columns of $\mathbf{T}_2$, we have $\sigma^2\mathbf{M}_E\mathbf{M}_B = [\mathbf{I}_{n..} - \mathbf{T}_2\mathbf{G}\mathbf{T}_2^t]\mathbf{T}_1\mathbf{A}^{-1}\mathbf{T}_1^t = 0$ which implies the independence between SSB/σ^2 and SSE/σ^2 when $\sigma_D^2 = 0$. Letting $MSB = SSB/rank(\mathbf{M}_B)$ and $MSE = SSE/rank(\mathbf{M}_E)$ and noting that $rank(\mathbf{M}_B) = b-1$ and that $rank(\mathbf{M}_E) = n.. - ab$, we conclude that

$$F = MSB/MSE \sim F_{[(b-1),(n..-ab)]} \tag{4}$$

where $F_{[q,p]}$ denotes the F distribution with q degrees of freedom in the numerator and p degrees of freedom in the denominator.

3 Tests based on Öfversten transformations

The basic idea behind the technique proposed by Öfversten (1993) to test for null variance components is to consider Householder orthogonal transformations of the data that generate independent quadratic forms that allow us to construct exact F tests even for unbalanced data. Öfversten (1993) considered models of the form (3), with $\mathbf{b} = [\mathbf{b}_1^t, \ldots, \mathbf{b}_k^t]^t$ with the additional requirement that the variance of $\mathbf{b}_i$ is $\sigma_i^2 \mathbf{I}_s$ and $\mathbf{Z} = [\mathbf{Z}_1, \ldots, \mathbf{Z}_k]$ is such that each row of $\mathbf{Z}_i$ has a single element equal to one and the others equal to zero.

Under the *UP* model, the specification matrix for the random effects $\mathbf{b}$ is partitioned as $\mathbf{Z} = [\mathbf{Z}_1,\ \mathbf{Z}_2]$, where $\mathbf{Z}_1$ corresponds to the random factor B ($\mathbf{b}_1$), and $\mathbf{Z}_2$ corresponds to the random interaction AB ($\mathbf{b}_2$). Since $\mathbf{Z}_1 = \mathbf{Z}_2(\mathbf{1}_a \otimes \mathbf{I}_b)$, the vector space generated by the columns of $\mathbf{Z}_1$ is a subspace of the vector space generated by the columns of $\mathbf{Z}_2$. Consequently the random effects $\mathbf{b}_1$ are nested within the random effects $\mathbf{b}_2$ and thus we may use the methodology developed in Öfversten (1993, Section 4) to obtain exact F tests for the hypotheses $\sigma_{\alpha B}^2 = 0$ and $\sigma_B^2 = 0$. As a matter of fact, the test obtained using Öfversten´s methodology does not coincide with $MSA/MSAB$, due to the latter does not have an exact F distribution for unbalanced data.

Under the *CP* model, the assumptions required in Öfversten (1993) do not hold, since the random effects in $\mathbf{b}_2$ are not independent and $\mathbf{Z}_2$ has rows with elements different from 1 or 0. However, note that $[\mathbf{X},\ \mathbf{Z}_1,\ \mathbf{Z}_2]$ is a matrix of dimension $(n_{\cdot\cdot} \times (ab+a))$ with

$$rank(\mathbf{X}) = a,\ rank[\mathbf{X},\ \mathbf{Z}_1] = a + (b-1)$$

and

$$rank[\mathbf{X},\ \mathbf{Z}_1,\ \mathbf{Z}_2] = a + (b-1) + (a-1)(b-1),$$

since $n_{ij} > 0$ for $i = 1, \ldots, a$ and $j = 1, \ldots, b$. Therefore, there exists an orthogonal $(n_{\cdot\cdot} \times n_{\cdot\cdot})$ matrix $\mathbf{C}$, such that

$$\mathbf{C}[\mathbf{X},\ \mathbf{Z}_1,\ \mathbf{Z}_2] = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{0} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

with $\mathbf{R}_{11}$, $\mathbf{R}_{22}$ e $\mathbf{R}_{33}$ respectively denoting $a \times a$, $(b-1) \times b$ and $(a-1)(b-1) \times (a-1)b$ full row rank upper triangular matrices and $\mathbf{R}_{12}$, $\mathbf{R}_{13}$ and $\mathbf{R}_{23}$ being conformable matrices. Such a $\mathbf{C}$ matrix may be constructed from successive Householder transformations applied to $[\mathbf{X},\ \mathbf{Z}_1,\ \mathbf{Z}_2]$ (see Rao and Mitra (1971, Sec. 11.5.2) and the Appendix for details). Pre-multiplication by $\mathbf{C}$, transforms model (3) into

$$\mathbf{CY} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \\ \mathbf{t}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{0} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} + \mathbf{CE},$$

where $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$ and $\mathbf{t}_4$ are column vectors of dimensions a, $b-1$, $(a-1)(b-1)$ and $n_{..} - ab$ respectively. Then,

$$\begin{bmatrix} \mathbf{t}_3 \\ \mathbf{t}_4 \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \sigma^2_{\tau D}\mathbf{R}_{33}[(\mathbf{I}_{a-1} - a^{-1}\mathbf{J}_{a-1}) \otimes \mathbf{I}_b]\mathbf{R}^t_{33} + \sigma^2\mathbf{I}_{(a-1)(b-1)} & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{I}_{n_{..}-ab} \end{bmatrix}\right)$$

and we can test $H_0 : \sigma^2_{\tau D} = 0$ using

$$F = \frac{\mathbf{t}^t_3\mathbf{t}_3/(a-1)(b-1)}{\mathbf{t}^t_4\mathbf{t}_4/(n_{..} - ab)}, \tag{5}$$

which follows an exact F distribution with $(a-1)(b-1)$ and $n_{..} - ab$ degrees of freedom under H_0.

Some additional steps are required to obtain the exact F test for $\sigma^2_D = 0$. First, note that both (3) and $\mathbf{ZVZ}^t$ are not affected by changing the positions of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ or of $\mathbf{b}_1$ and $\mathbf{b}_2$; then rewrite (3) as

$$\mathbf{Y} = [\mathbf{X}\ \mathbf{Z}_2\ \mathbf{Z}_1] \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{b}_2 \\ \mathbf{b}_1 \end{bmatrix} + \mathbf{E}. \tag{6}$$

Now observe that

$$rank(\mathbf{X}) = a, \quad rank[\mathbf{X} \mid \mathbf{Z}_2] = a + (a-1)(b-1),$$

and

$$rank[\mathbf{X} \mid \mathbf{Z}_2 \mid \mathbf{Z}_1] = a + (a-1)(b-1) + (b-1)$$

so we may follow the same lines of the argument used to construct a test for $\sigma^2_{\tau D} = 0$. More specifically, there exists an orthogonal matrix $\mathbf{C}^*$ such that

$$\mathbf{C}^*[\mathbf{X}\ \mathbf{Z}_2\ \mathbf{Z}_1] = \begin{bmatrix} \mathbf{R}^*_{11} & \mathbf{R}^*_{12} & \mathbf{R}^*_{13} \\ \mathbf{0} & \mathbf{R}^*_{22} & \mathbf{R}^*_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}^*_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{R}^*_{11}$, $\mathbf{R}^*_{22}$ and $\mathbf{R}^*_{33}$ are respectively, $a \times a$, $(a-1)(b-1) \times (a-1)b$ and $(b-1) \times b$ full row rank upper triangular matrices and $\mathbf{R}^*_{12}$, $\mathbf{R}^*_{13}$ and $\mathbf{R}^*_{23}$ are conformable matrices. Pre-multiplication by $\mathbf{C}^*$ transforms model (6) into

$$\mathbf{C}^*\mathbf{Y} = \begin{bmatrix} \mathbf{t}^*_1 \\ \mathbf{t}^*_2 \\ \mathbf{t}^*_3 \\ \mathbf{t}^*_4 \end{bmatrix} = \begin{bmatrix} \mathbf{R}^*_{11} & \mathbf{R}^*_{12} & \mathbf{R}^*_{13} \\ \mathbf{0} & \mathbf{R}^*_{22} & \mathbf{R}^*_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}^*_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{b}_2 \\ \mathbf{b}_1 \end{bmatrix} + \mathbf{C}^*\mathbf{E}.$$

Therefore, we have

$$\begin{bmatrix} \mathbf{t}^*_3 \\ \mathbf{t}^*_4 \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} \sigma^2_D \mathbf{R}^*_{33}\mathbf{R}^{*t}_{33} + \sigma^2 \mathbf{I}_{b-1} & \mathbf{0} \\ \mathbf{0} & \sigma^2 \mathbf{I}_{n_{\cdot\cdot}-ab} \end{bmatrix}\right),$$

and thus, we may test the null hypothesis $H_0 : \sigma^2_D = 0$ using the statistic

$$F = \frac{\mathbf{t}^{*t}_3 \mathbf{t}^*_3/(b-1)}{\mathbf{t}^{*t}_4 \mathbf{t}^*_4/(n_{\cdot\cdot} - ab)}, \tag{7}$$

that follows an exact F distribution with $b-1$ and $n_{\cdot\cdot} - ab$ degrees of freedom under H_0. This statistic agrees with the statistic (4) obtained from $ANOVA$ tables using Type III sums of squares.

To illustrate the results, we consider an study conducted to investigate the effects of spools and of their position (left or right) in a spinning machine on the tension of synthetic filaments in a textile plant. Differences in tension may affect the dye affinity of the filaments and thus jeopardize the quality of the fabric produced therewith. The tension of the filaments from each of 12 spools selected from a very large batch was observed haphazardly at different instants of the production process on both sides of the spinning machine. Position is viewed as the fixed factor and spool as the random factor. The data are displayed in Table 1.

The required model specification matrices are

Table 1: Tension of synthetic filaments (cN)

Position	Spool (B)											
(A)	1	2	3	4	5	6	7	8	9	10	11	12
Left (1)	12.2	17.3	14.7	14.8	15.5	15.3	16.1	16.1	15.0	15.4	15.3	15.5
	12.1	17.6	15.7	14.7	15.4	15.4	16.7	16.0	14.8	15.3	15.2	15.9
	12.0			14.4	15.5	15.1	16.2	16.0	14.9	15.3	15.2	15.3
	11.8			14.6			16.5	16.1	14.8	15.0		
Right (2)	17.7	18.4	18.0	17.6	19.0	19.7	17.9	19.1	20.7	19.6	18.4	20.6
	17.6	18.4	19.7	17.5	18.9	19.6	18.0	18.9	20.4	20.4	18.4	20.0
	17.8		18.5		18.9	18.9	18.0	19.1		18.8	18.4	20.1
	17.8					19.5	18.2			23.0	18.4	

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_{40} & \mathbf{1}_{38} \\ \mathbf{1}_{40} & -\mathbf{1}_{38} \end{bmatrix}^t \quad \mathbf{Z}_1 = \begin{bmatrix} \bigoplus_{j=1}^{12} \mathbf{1}_{n_{1j}} \\ \bigoplus_{j=1}^{12} \mathbf{1}_{n_{2j}} \end{bmatrix} \quad \text{and} \quad \mathbf{Z}_2 = \begin{bmatrix} \bigoplus_{j=1}^{12} \mathbf{1}_{n_{1j}} \\ \bigoplus_{j=1}^{12} -\mathbf{1}_{n_{2j}} \end{bmatrix}$$

An application of the orthogonal transformation defined by $\mathbf{C}$ to the data $\mathbf{Y}$ generates the following vectors required to construct the statistic (5) used to test $H_0 : \sigma^2_{\tau D} = 0$

$$\mathbf{t}_3 = [\ -2.89 \quad -2.67 \quad 0.19 \quad 0.96 \quad 0.43 \quad -0.46 \quad -3.36 \quad 1.93 \quad -1.34 \quad -1.48 \quad 1.34\ ]^t$$

and

$$\mathbf{t}_4 = [\ \mathbf{t}_{41}^t \quad \mathbf{t}_{42}^t \quad \mathbf{t}_{43}^t \quad \mathbf{t}_{44}^t \quad \mathbf{t}_{45}^t\ ]^t$$

with

$$\begin{array}{lcl} \mathbf{t}_{41} & = & [-0.08 \quad 0.02 \quad 0.07 \quad -0.13 \quad -0.03 \quad -0.13 \quad 0.23 \quad 0.13 \quad 0.13 \quad -0.17 \quad 0.12]^t \\ \mathbf{t}_{42} & = & [\ 0.02 \quad 0.02 \quad 0.06 \quad 0.46 \quad -0.14 \quad -0.01 \quad -0.11 \quad 0.09 \quad 0.09 \quad 0.14 \quad 0.14]^t \\ \mathbf{t}_{43} & = & [-0.62 \quad 1.08 \quad -0.12 \quad 0.19 \quad 0.09 \quad 0.44 \quad 0.34 \quad 0.34 \quad 0.17 \quad 0.07 \quad -0.63]^t \\ \mathbf{t}_{44} & = & [-0.03 \quad -0.22 \quad -0.12 \quad -0.12 \quad 0.08 \quad 0.08 \quad -0.12 \quad 0.08 \quad -0.11 \quad -0.41 \quad -0.98]^t \\ \mathbf{t}_{45} & = & [-0.18 \quad -1.78 \quad 2.42 \quad -0.10 \quad -0.10 \quad -0.10 \quad -0.10 \quad 0.25 \quad -0.35 \quad -0.25]^t. \end{array}$$

Similarly, an application of the orthogonal transformation $\mathbf{C}^*$ to the data $\mathbf{Y}$ yields the vectors required to construct the statistics (7) proposed to test $H_0 : \sigma^2_D = 0$, namely

$$\mathbf{t}_3^* = [\ -6.56 \quad -1.23 \quad -0.74 \quad -2.98 \quad -0.62 \quad -0.35 \quad 0.91 \quad 0.05 \quad -0.39 \quad -1.17 \quad 1.94\ ]^t$$

and

$$\mathbf{t}_4^* = [\ \mathbf{t}_{41}^{*t} \quad \mathbf{t}_{42}^{*t} \quad \mathbf{t}_{43}^{*t} \quad \mathbf{t}_{44}^{*t} \quad \mathbf{t}_{45}^{*t}\ ]^t$$

with

$$
\begin{array}{lcrrrrrrrrrrr}
\mathbf{t}^*_{41} & = & [0.09 & 0.19 & 0.03 & -0.17 & -0.07 & -0.17 & 0.12 & 0.02 & 0.02 & -0.28 & 0.10]^t \\
\mathbf{t}^*_{42} & = & [0.00 & 0.00 & 0.05 & 0.45 & -0.15 & -0.06 & -0.16 & 0.04 & 0.04 & -0.16 & -0.16]^t \\
\mathbf{t}^*_{43} & = & [-0.87 & 0.83 & -0.37 & -0.11 & -0.21 & -0.33 & -0.43 & -0.43 & 0.36 & 0.26 & -0.44]^t \\
\mathbf{t}^*_{44} & = & [0.16 & -0.05 & 0.05 & 0.05 & 0.25 & 0.25 & 0.05 & 0.25 & 0.31 & 0.01 & -0.85]^t \\
\mathbf{t}^*_{45} & = & [-0.05 & -1.65 & 2.55 & 0.06 & 0.06 & 0.06 & 0.06 & 0.45 & -0.15 & -0.05]^t. &
\end{array}
$$

The results suggest that both the Spool effect $[F_{obs} = (60.6/11)/(13.6/54) = 21.9, p < 0.0001]$ and the Position × Spool interaction $[F_{obs} = (37.6/11)/(13.6/54) = 13.6, p < 0.0001]$ are highly significant. Under the *UP* model, the usual test for $\sigma^2_B = 0$ does not reject the null hipothesis $[F_{obs} = (61.7/11)/(38.1/11) = 1.62, p = 0.2187]$.

4 Power of the exact test

It is well known (see Khuri et al. (1998), for example) that the standard F-tests from $ANOVA$ tables obtained under mixed models for balanced data are uniformly most powerful unbiased ($UMPU$), uniformly most powerful invariant ($UMPI$) and uniformly most powerful invariant unbiased ($UMPIU$). Unfortunately, this is not always true for unbalanced data. In such cases, although we cannot obtain optimum tests, we can use the results of Öfversten (1993) to set lower bounds for the power of the tests derived in Section 3.

Using the same arguments as in Lemma 6 of Öfversten (1993), we obtain the following lower bound for the power of the test statistic (5):

$$1 - F_{[(a-1)(b-1), n_{\cdot\cdot} - ab]}\left(\frac{c}{\delta(\sigma^2_{\tau D}/\sigma^2) + 1}\right). \tag{8}$$

Here, δ is the minimum eigenvalue of $\mathbf{R}_{33}[(\mathbf{I}_{a-1} - a^{-1}\mathbf{J}_{a-1}) \otimes \mathbf{I}_b]\mathbf{R}^t_{33}$ and c is the critical value corresponding to a test of size α, i.e., $\alpha = 1 - F_{[(a-1)(b-1), n_{\cdot\cdot} - ab]}(c)$. Details are presented in the Appendix. For the statistic (7), the lower bound is computed similarly and is given by

$$1 - F_{[(b-1), n_{\cdot\cdot} - ab]}\left(\frac{c}{\delta^*(\sigma^2_D/\sigma^2) + 1}\right), \tag{9}$$

with δ^* denoting the minimum eigenvalue of $\mathbf{R}^*_{33}\mathbf{R}^{*t}_{33}$.

To evaluate the loss in the power of the statistic (7) for testing $\sigma_D^2 = 0$ with unbalanced data, we consider a limited simulation study. The data are generated according to a factorial design with a fixed factor A with 2 levels, a random factor B with 3 levels and n_{ij} replicates under each combination of level i of factor A and level j of factor B. Following Khuri (1987), we use

$$\phi = \frac{1}{1 + \chi^2/n_{..}},$$

with $\chi^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}(n_{ij} - \bar{n}_{..})^2/\bar{n}_{..}$ and $\bar{n}_{..} = n_{..}/ab$ as a measure of the degree of imbalance. It is easy to see that $0 < \phi \leq 1$ and that the upper limit is attained if and only if the data is balanced. Large values of χ^2, and hence small values of ϕ correspond to a high degree of imbalance. In Table 2 we show the number of observations under each combination of the levels of factors A and B for different patterns of imbalance, i.e. different levels of ϕ.

Table 2: Number of replicates under level i of factor A and level j of factor B considered in the simulation

	Number of observations					$n_{..} = 18$
ϕ	n_{11}	n_{12}	n_{13}	n_{21}	n_{22}	n_{23}
1	3	3	3	3	3	3
0,96	3	3	4	2	3	3
0,75	5	4	5	1	1	2
0,51	1	1	8	6	1	1
0,31	1	1	1	1	1	13
	Number of observations					$n_{..} = 36$
1	6	6	6	6	6	6
0,93	4	8	4	6	6	8
0,77	1	9	2	7	8	9
0,48	1	10	2	1	4	18
0,32	1	1	25	7	1	1

We replicate the design detailed in Table 2 using the *CP* model to generate 1.000 samples under each combination of the following values for the parameters σ^2, σ_D^2, $\sigma_{\tau D}^2$ and $n_{..}$:

- $\sigma^2 = 1$, $\sigma^2_{\tau D} = 4$ or 12 and $\sigma^2_D = 0, 1, 4, 9$ or 16

- $n_{..} = 18$ or 36.

The results of the simulation are summarized in Tables 3 and 4. Within parentheses, we show the lower limits for the power of the test under each pattern of imbalance and different levels of σ^2_D. The first column corresponds to balanced data ($\phi = 1$). Since the F test under this condition is optimum, this is where we achieve the highest power. In addition to that, for $\sigma^2_D > 1$ and unbalanced data ($\phi < 1$) we also observe that the power is not much less than the power of the exact test traditionally used for balanced data under similar conditions.

Table 3: Observed power for the exact F test of $H_0 : \sigma^2_D = 0$, when $n_{..} = 18$

	Imbalance Measure				
$\sigma^2 = 1$, $\sigma^2_{\tau D} = 4$	$\phi = 1$	$\phi = 0.96$	$\phi = 0.75$	$\phi = 0.51$	$\phi = 0.31$
$\sigma^2_D = 0$	0.057	0.049	0.043	0.056	0.068
1	0.642	0.596 (0.551)*	0.498 (0.428)	0.433 (0.344)	0.408 (0.310)
4	0.937	0.938 (0.839)	0.898 (0.763)	0.841 (0.694)	0.816 (0.659)
9	0.995	0.995 (0.923)	0.984 (0.881)	0.974 (0.840)	0.974 (0.818)
16	1.00	1.00 (0.955)	1.00 (0.930)	0.997 (0.904)	1.00 (0.890)
$\sigma^2 = 1$, $\sigma^2_{\tau D} = 16$	$\phi = 1$	$\phi = 0.96$	$\phi = 0.75$	$\phi = 0.51$	$\phi = 0.31$
$\sigma^2_D = 0$	0.055	0.039	0.052	0.037	0.042
1	0.609	0.597 (0.551)	0.530 (0.428)	0.399 (0.344)	0.411 (0.310)
4	0.941	0.946 (0.839)	0.901 (0.763)	0.844 (0.694)	0.824 (0.659)
9	0.996	0.995 (0.923)	0.990 (0.881)	0.982 (0.840)	0.969 (0.818)
16	1.00	0.999 (0.955)	0.998 (0.930)	0.999 (0.904)	0.997 (0.890)

$*$ Lower limits for the power obtained from expression(8)

The lower limit for the power decreases with the degree of imbalance (ϕ), since the minimum eigenvalue of $\mathbf{R}_{33}\mathbf{R}^t_{33}$ also decreases with ϕ (see Table 5).

Table 4: Observed power for the exact F test of $H_0 : \sigma^2_D = 0$, when $n_{..} = 36$

$\sigma^2 = 1$, $\sigma^2_{\tau D} = 4$	Imbalance Measure				
	$\phi = 1$	$\phi = 0.93$	$\phi = 0.77$	$\phi = 0.48$	$\phi = 0.32$
$\sigma^2_D = 0$	0.040	0.041	0.059	0.055	0.054
1	0.786	0.768 (0.743)*	0.669 (0.543)	0.617 (0.418)	0.460 (0.384)
4	0.983	0.983 (0.923)	0.961 (0.835)	0.956 (0.756)	0.880 (0.730)
9	1.00	1.00 (0.964)	0.999 (0.920)	0.999 (0.877)	0.984 (0.862)
16	1.00	1.00 (0.980)	1.00 (0.954)	1.00 (0.928)	1.00 (0.918)
$\sigma^2 = 1$, $\sigma^2_{\tau D} = 16$	$\phi = 1$	$\phi = 0.93$	$\phi = 0.77$	$\phi = 0.48$	$\phi = 0.32$
$\sigma^2_D = 0$	0.052	0.055	0.048	0.046	0.060
1	0.793	0.784 (0.743)	0.676 (0.543)	0.597 (0.418)	0.471 (0.384)
4	0.991	0.988 (0.923)	0.964 (0.835)	0.943 (0.756)	0.890 (0.730)
9	1.00	1.00 (0.964)	0.999 (0.920)	0.997 (0.877)	0.992 (0.862)
16	1.00	1.00 (0.980)	1.00 (0.954)	1.00 (0.928)	1.00 (0.918)

$*$ Lower limits for the power obtained from expression (8)

As expected, the observed power also decreases with the degree of imbalance (ϕ); however, the loss in power (with respect to the balanced data set of the same size) seems to be acceptable. In fact, the loss of observed power was never greater than 15% for all values of $\sigma^2_D > 1$. In the light of the lower bound computations, the maximum loss cannot exceed 30%.

Note that for $\sigma^2_D = 1$, the variance of D_j coincides with the error variance and in such a case it is reasonable to have little power to test for $\sigma^2_D = 0$. To see this, observe that σ^2_D

Table 5: Minimum eigenvalues of $\mathbf{R}_{33}\mathbf{R}^t_{33}$ for the numbers of replicates shown in Table 2

$n_{..}$	18					36				
ϕ	1	0.96	0.75	0.51	0.31	1	0.93	0.77	0.48	0.32
δ^*	6.00	5.21	3.26	2.33	2.00	12.00	10.05	4.32	2.69	2.35

is in the denominator of (9). The observed power increases with σ_D^2 and with $n_{..}$, since the minimum eigenvalues δ^* in the denominator of (9) increases with $n_{..}$. On the other hand, we do not observe changes in the power as a function of the values of $\sigma_{\tau D}^2$ and the lower bound does not depend on that either.

5 Conclusion

We have shown that MSB/MSE (based on Type III sums of squares) still has an exact F distribution when $\sigma_D^2 = 0$ under unbalanced data situations. We have also shown that the assumptions in Öfversten (1993) may be relaxed to generate exact F tests for mixed models with dependent random errors as in the CP model for two-way layouts. For such a model, the standard hypotheses of no random factor main effect and no interaction may be easily specified and the corresponding exact tests may be obtained even for unbalanced data. Such tests do not have optimal properties as their counterparts for balanced data, but their power lies within reasonable limits for practical applications. From a practical point of view, Type III sums of squares that appear in most commercial statistical software packages may be considered to test for the nonexistence of the random factor main effects in the presence of interaction.

Acknowledgements

The authors wish to thank the editor and the referee for their thoughtful suggestions. They are also grateful to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Brazil, Consejo de Investigaciones de la Universidad Nacional de Tucumán (CIUNT), Argentina and to the National Institutes of Health (NIH-PHS-R01-HD36848), USA for financial support.

References

Crainiceanu, C.M. and Ruppert, D. (2004). Likelihood ratio tests in linear mixed models with one variance component. *Journal of the Royal Statistical Society, B*,

66, 165-185.

Hinkelmann, K. (2000). Resolving the Mixed Models Controversy, Comments. *The American Statistician*, **54**, 228.

Hinkelmann, K. and Kempthorne, O. (1994). *Design and Analysis of Experiments: Introduction to Experimental Design, Vol 1.* New York: Wiley.

Hocking, R.R. (1973). A discussion of the two-way mixed model. *The American Statistician*, **27**, 148-152.

Khuri, A.I. (1987). Measures of imbalance for unbalanced models. *Biometrical Journal*, **29**, 383-396.

Khuri, A.I., Mathew, T. and Sinha, B.K. (1998). *Statistical Test for Mixed Linear Models.* New York: Wiley.

Kshirsagar, A.M. (1983). *A Course in Linear Models.* New York: Dekker.

Lencina, V.B., Singer, J.M and Stanek, E.J. (2005). Much ado about nothing: the mixed models controversy revisited. *International Statistical Review*, **73**, 9-20.

Milliken, G.A. and Johnson, D.E. (1984). *Analysis of Messy Data: Vol. 1. Designed Experiments.* Belmont, CA: Lifetime Learning Publications.

Montgomery, D.C. (1997). *Design and Analysis of Experiments, 4th edition.* New York: Wiley.

Neter, J., Wasserman, W., Kutner, M.H. and Nachtsheim, C.J.(1996). *Applied Linear Statistical Models (4th ed.).* Homewood, Ill: Irwin.

Öfversten, J. (1993). Exact Test for Variance Components in Unbalanced Mixed Linear Model. *Biometrics*, **49**, 45-57.

Rao, C.R. and Mitra, S.K. (1971). *Generalized Inverse of Matrices and its Applications.* New York: Wiley.

SAS Institute. Inc. (1990). *SAS Users Guide: Statistics (Version 6).* Cary, NC.

SEARLE, S.R. (1971). *Linear Models*. New York: Wiley.

SEARLE, S.R. (1982). *Matrix Algebra Useful for Statistics*. New York: Wiley.

STRAM, D.O. AND LEE, J.W. (1994). Variance Components Testing in the Longitudinal Mixed Effects Model. *Biometrics*, **50**, 1171-1177.

VOSS, D.T. (1999). Resolving the Mixed Models Controversy. *The American Statistician*, **53**, 352-356.

VOSS, D.T. (2000). Resolving the Mixed Models Controversy, Reply. *The American Statistician*, **54**, 229-230.

WOLFINGER, R. AND STROUP, W.W. (2000). Resolving the Mixed Models Controversy, Comments. *The American Statistician*, **54**, 228.

Appendix

Computation of the C and $\mathbf{C}^*$ matrices

Let $(\mathbf{a})_k$ or a_k denote the k-th component of a vector $\mathbf{a}$. Also, for a vector (matrix) $\mathbf{a} \in \mathbf{R}^n$ ($\mathbf{A} \in \mathbf{R}^{n\times m}$), let $\mathbf{a}^{(j)}$ ($\mathbf{A}^{(j)}$) denote a new vector (matrix) obtained from $\mathbf{a}$ ($\mathbf{A}$) excluding the first j components (rows), i.e., $\mathbf{a}^{(j)} = (a_{j+1} \cdots a_n)^t \in \mathbf{R}^{n-j}$, and $\mathbf{A}^{(j)} = (\mathbf{a}_1^{(j)} \quad \mathbf{a}_2^{(j)} \cdots \mathbf{a}_m^{(j)}) \in \mathbf{R}^{n-j\times m}$. Let $\mathbf{M} = [\mathbf{X},\ \mathbf{Z}_1,\ \mathbf{Z}_2]$ be as defined in Section 3. The algorithm to compute $\mathbf{C}$ and $\mathbf{C}[\mathbf{X},\ \mathbf{Z}_1,\ \mathbf{Z}_2]$ is

- ***Let*** $j = 0$, $i = 1$, $\mathbf{C} = \mathbf{I}_{n..}$.
- ***While*** $\mathrm{M}^{(j)} \neq \mathbf{0}$ and the number of rows of $\mathbf{M}^{(j)}$ is greater than 1 ***do***:

  {

  - ***Select*** i, ($i \leq m$) such that, for all $i' < i$,

$$\mathbf{m}_{i'}^{(j)} = \mathbf{0} \quad \text{and} \quad \mathbf{m}_i^{(j)} \neq \mathbf{0},$$

– ***Construct*** $\mathbf{H}$ for $\mathbf{m}_i^{(j)}$, such that $\mathbf{H}\mathbf{m}_i^{(j)} = (\lambda, 0, \cdots, 0)^t$, setting $\mathbf{H} = \mathbf{I} - \mathbf{h}\mathbf{h}^t$ with

$$\begin{cases} \lambda = -((\mathbf{m}_i^{(j)})_1/|(\mathbf{m}_i^{(j)})_1|)\sqrt{(\mathbf{m}_i^{(j)})^t\mathbf{m}_i^{(j)}} \\ h_1 = \sqrt{1/2(1-(\mathbf{m}_i^{(j)})_1/\lambda)} \\ h_k = -(\mathbf{m}_i^{(j)})_k/(2h_1\lambda) \quad \text{for} \quad k = 2, \ldots, n_{\cdot\cdot} - j. \end{cases}$$

– ***Assign*** $\begin{pmatrix} \mathbf{I}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix} \mathbf{C}$ to $\mathbf{C}$,

– ***Assign*** $j+1$ to j,

– ***Assign*** $\mathbf{CM}$ to $\mathbf{M}$.

}

- ***End***.

The resulting $\mathbf{C}$ and $\mathbf{M} = \mathbf{C}[\mathbf{X},\ \mathbf{Z}_1,\ \mathbf{Z}_2]$ are the required matrices. The algorithm to compute $\mathbf{C}^*$ and $\mathbf{M} = \mathbf{C}^*[\mathbf{X},\ \mathbf{Z}_2,\ \mathbf{Z}_1]$ follows the same steps.

Lower bound for the power of the exact F-test

Since $\mathbf{R}_{33}$ is a $(b-1)\times b$ full row rank matrix, it follows that $\mathbf{R}_{33}[(\mathbf{I}_{a-1}-a^{-1}\mathbf{J}_{a-1})\otimes\mathbf{I}_b]\mathbf{R}_{33}^t$ is a $(b-1)\times(b-1)$ symmetric, positive definite matrix. Therefore there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}\mathbf{R}_{33}[(\mathbf{I}_{a-1} - a^{-1}\mathbf{J}_{a-1}) \otimes \mathbf{I}_b]\mathbf{R}_{33}^t\mathbf{P}^t = \mathbf{\Delta},$$

where $\mathbf{\Delta}$ is a diagonal matrix with the eigenvalues of $\mathbf{R}_{33}[(\mathbf{I}_{a-1} - a^{-1}\mathbf{J}_{a-1}) \otimes \mathbf{I}_b]\mathbf{R}_{33}^t$ along the main diagonal. Letting δ be the minimum eigenvalue, we may follow the proof of Lemma 6 in Öfversten (1993) to obtain the lower bound for the power of (5) as a test statistic for the hypothesis $\sigma_{\tau D}^2 = 0$. A similar argument may be employed for the hypothesis $\sigma_D^2 = 0$ if we use the statistic (7).

Résumé

Dans le contexte des modèles avec un facteur fixé et un facteur aléatoire qui genèrent la bien-connue controverse de modèles mixtes, nous considérons des testes F pour l'hypothèse

que, en présence de l'intéraction, l'effet du facteur aléatoire est nul. Nous montrons que, dans le modèle à paramètres restringés (CP), MSB/MSE (employant la notation usuelle de ANOVA) suit une distribution exacte F quand l'hypothèse nulle est vraie, même dans des situations avec des données non-équilibrées. Par addition, nous obtenons un test exact F pour ce qui est en genéral (et par erreur) consideré une hypothèse equivalente dans le modèle à paramètres non-restringés (UP). Dans le cas des données non-équilibrées, la statistique de test proposée ne coincide pas avec $MSB/MSAB$, qui est la statistique usuellement recommendée. Nous calculons la puissanece du test exact sous différents padrons de deséquilibre et nous montrons que même si la perte de puissance augmente avec le degré de deséquilibre, elle se mantient acceptable du point de vue practique.