Further Investigation of alternative Formulation of RP Model with Response Error

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1. INTRODUCTION

We explore the predictor that will result in a simple random sample with response error when a different model is postulated. This is a follow-up to c06ed06.doc where we try to come up with an alternative approach to the partially conditional approach. The model we describe here is closely related to the model described in X. Director for the Future (page 5-1 etc in the Mendosa workbook).

1.1. Potentially observable Response and the Population

Consider the setting where for subject $s$, $s=1,\ldots,N$, we may potentially observe response $y_{st}$, where $t=1,2,\ldots,M$. Notice that the response is non-stochastic. Let us define

$$y_s = \frac{1}{M} \sum_{t=1}^{M} y_{st}$$

for $s=1,\ldots,N$, and

$$\mu = \frac{1}{N} \sum_{s=1}^{N} y_s.$$ 

Using these terms, we define $w_{st} = y_{st} - y_s$ and $\beta_s = y_s - \mu$ so that

$$y_{st} = y_s + w_{st} = \mu + \beta_s + w_{st}.$$ 

We refer to $w_{st}$ as response error, even though the subscript $t=1,2,\ldots,M$ may be an index of different conditions, not an “error process”. In addition define $\sigma^2 = \frac{1}{N-1} \sum_{s=1}^{N} (y_s - \mu)^2$ and

$$\sigma_s^2 = \frac{1}{M-1} \sum_{t=1}^{M} w_{st}^2 \text{ for } s=1,\ldots,N.$$ 

1.2 Selection of Response

Let us define $U_{jt}^{(s)}$ as an indicator of response $t$ on selection $j$ for subject $s$, and define a randomly selected response error as $W_{sj} = \sum_{t=1}^{M} U_{jt}^{(s)} w_{st}$. We use $\xi_2$ to index expectation with respect to response error. We assume that $E_{\xi_2} (U_{jt}^{(s)}) = \frac{1}{M}$ and that sampling of response error is with replacement. With this assumption,
\[
\text{var}_{\xi_j} \left( U_{jt}^{(s)} \right) = E_{\xi_j} \left[ \left( U_{jt}^{(s)} \right)^2 \right] - \left[ E_{\xi_j} \left( U_{jt}^{(s)} \right) \right]^2
\]
\[
= \frac{1}{M} - \left( \frac{1}{M} \right)^2,
\]
\[
= \frac{M - 1}{M^2}.
\]

Also,
\[
\text{cov}_{\xi_j} \left( U_{jt}^{(s)} , U_{jt}^{(s)} \right) = E_{\xi_j} \left[ U_{jt}^{(s)} U_{jt}^{(s)} \right] - \left[ E_{\xi_j} \left( U_{jt}^{(s)} \right) \right] \left[ E_{\xi_j} \left( U_{jt}^{(s)} \right) \right]
\]
\[
= \frac{1}{M^2} - \left( \frac{1}{M} \right)^2
\]
\[
= 0.
\]

and
\[
\text{cov}_{\xi_j} \left( U_{jt}^{(s)} , U_{jt^{*}}^{(s)} \right) = E_{\xi_j} \left[ U_{jt}^{(s)} U_{jt^{*}}^{(s)} \right] - \left[ E_{\xi_j} \left( U_{jt}^{(s)} \right) \right] \left[ E_{\xi_j} \left( U_{jt^{*}}^{(s)} \right) \right]
\]
\[
= \frac{1}{M^2} - \left( \frac{1}{M} \right)^2
\]
\[
= 0,
\]
so that \( \text{cov}_{\xi_j} \left( U_{jt}^{(s)} , U_{jt^{*}}^{(s)} \right) = 0 \) whenever \( j \neq j^* \). These assumptions imply that
\[
E_{\xi_j} \left( W_{sj} \right) = E_{\xi_j} \left( \sum_{t=1}^{M} U_{jt}^{(s)} w_{st} \right)
\]
\[
= \sum_{t=1}^{M} E_{\xi_j} \left( U_{jt}^{(s)} \right) w_{st}
\]
\[
= \sum_{t=1}^{M} \frac{1}{M} w_{st}
\]
\[
= 0.
\]

Also,
\[
\text{var}_{\xi_j} \left( W_{sj} \right) = \text{var}_{\xi_j} \left( \sum_{t=1}^{M} U_{jt}^{(s)} w_{st} \right)
\]
\[
= \sum_{t=1}^{M} \text{var}_{\xi_j} \left( U_{jt}^{(s)} \right) w_{st}^2
\]
\[
= \left( \frac{M - 1}{M} \right) \left( \frac{1}{M - 1} \sum_{t=1}^{M} w_{st}^2 \right)
\]
\[
= \left( \frac{M - 1}{M} \right)^2 \sigma_s^2.
\]

Also, when \( j \neq j^* \)
\[
\text{cov}(W_{sj}, W_{sj^*}) = E_{\mathcal{E}_2} \left[ W_{sj} W_{sj^*} \right] \\
= E_{\mathcal{E}_2} \left[ \left( \sum_{t=1}^{M} U_{jt}^{(s)} W_{st} \right) \left( \sum_{t=1}^{M} U_{jt^*}^{(s)} W_{st^*} \right) \right] \\
= E_{\mathcal{E}_2} \left[ \left( \sum_{t=1}^{M} U_{jt}^{(s)} U_{jt}^{(s^*)} W_{st}^2 \right) + \left( \sum_{t=1}^{M} \sum_{t^*}^{M} U_{jt}^{(s)} U_{jt}^{(s^*)} W_{st} W_{st^*} \right) \right] \\
= \sum_{t=1}^{M} E_{\mathcal{E}_2} \left( U_{jt}^{(s)} U_{jt}^{(s^*)} \right) W_{st}^2 + \sum_{t=1}^{M} \sum_{t^*}^{M} E_{\mathcal{E}_2} \left( U_{jt}^{(s)} U_{jt}^{(s^*)} \right) W_{st} W_{st^*} 
\]

Now \( E_{\mathcal{E}_2} \left( U_{jt}^{(s)} U_{jt}^{(s^*)} \right) = \frac{1}{M^2} \) and \( E_{\mathcal{E}_2} \left( U_{jt}^{(s)} U_{jt}^{(s^*)} \right) = \frac{1}{M^2} \). As a result,

\[
\text{cov}(W_{sj}, W_{sj^*}) = \frac{1}{M^2} \left[ \sum_{t=1}^{M} W_{st}^2 + \sum_{t=1}^{M} \sum_{t^*}^{M} W_{st} W_{st^*} \right] \\
= \frac{1}{M^2} \left[ \sum_{t=1}^{M} W_{st} \left( \sum_{t^*}^{M} W_{st^*} \right) \right] \\
= 0
\]

for all \( s = 1, \ldots, N \); \( s^* = 1, \ldots, N \); \( j = 1, \ldots, r_s \); \( j^* (\neq j) = 1, \ldots, r_s \).

### 1.3 Summary Notation

We summarize notation for the problem. We represent the potentially observable response \( t \) by \( y_s \) for subject \( s \), \( s = 1, \ldots, N \), \( t = 1, 2, \ldots, M \). We define \( y_{s} = \frac{1}{M} \sum_{t=1}^{M} y_{st} \) and express \( y_{st} = y_s + w_{st} \), with additional parameters defined by \( \mu = \frac{1}{N} \sum_{s=1}^{N} y_s \), \( \beta_s = y_s - \mu \), \( \sigma^2 = \frac{1}{N-1} \sum_{s=1}^{N} (y_s - \mu)^2 \) and \( \sigma^2_s = \frac{1}{M-1} \sum_{t=1}^{M} w_{st}^2 \).

We define \( U_{jt}^{(s)} \) to be an indicator of response error \( t \) on selection \( j \) for subject \( s \), and \( W_{sj} = \sum_{t=1}^{M} U_{jt}^{(s)} w_{st} \). Using this random variable, we represent response on selection \( j \) for subject \( s \) by

\[
Y_{sj} = y_s + W_{sj} 
\]

We assume that \( E_{\mathcal{E}_2} \left( U_{jt}^{(s)} \right) = \frac{1}{M} \) and that sampling of response error is with replacement. As a result, \( E_{\mathcal{E}_2} \left( W_{sj} \right) = 0 \) for all \( s = 1, \ldots, N \), \( k = 1, \ldots, r_s \); \( \text{var}(W_{sj}) = \left( \frac{M-1}{M} \right) \sigma^2_s \) for all \( s = 1, \ldots, N \),
\[ j = 1, \ldots, r_s \] and \( \text{cov}(W_{sj}, W_{sj'}) = 0 \) for all \( j \neq j^* \) and \( s = 1, \ldots, N \). Finally, we assume \( r_s = 1 \) for all \( s = 1, \ldots, N \) so that \( j = 1 \).

2. RANDOM PERMUTATION OF THE POPULATION

Let us define an indicator random variable \( U_{is} \) that has the value of one if subject \( s \) is selected in position \( i \), and zero otherwise. Define \( U_i = (U_{i1} \ U_{i2} \ \cdots \ U_{iN})' \) and \( y = (y_1 \ y_2 \ \cdots \ y_N)' \). Also, let us define \( Y_R = (\left( Y_j \right)) = (Y_{1j} \ Y_{2j} \ \cdots \ Y_{nj})' \). We suppose that the subjects in the population are permuted. We can represent a random variable corresponding to position \( i \) in the permutation as \( Y_j = \sum_{s=1}^{N} U_{is} Y_{sj} \) or \( Y_j = U_i' Y_R \).

2.1. More Elementary Random Variables

An element in the sum \( \sum_{s=1}^{N} U_{is} Y_{sj} \) is given by

\[
U_{is} Y_{sj} = U_{is} Y_{sj} + U_{is} W_{sj}
\]

\[ = U_{is} Y_{sj} + U_{is} \sum_{t=1}^{M} U_{it}^{(t)} W_{st}.'
\]

These random variables are an expansion of the random variables which, when summed, add up to \( Y_j \). Notice that each of these random variables, when expressed as

\[
U_{is} Y_{sj} = \sum_{t=1}^{M} \left( U_{is} \frac{Y_{st}}{M} + U_{is} U_{it}^{(t)} W_{st} \right)
\]

is in turn a sum of additional random variables. The more elementary variables are given by

\[
U_{is} \frac{Y_{st}}{M} + U_{is} U_{it}^{(t)} W_{st}.
\]

The first random variable in this expression, when summed over \( s \) and \( t \), is the latent value for PSU \( i \). The second random variable corresponds to response error.

In this manuscript, we focus on alternative ways of defining a term similar to the second term, \( U_{is} U_{jt}^{(t)} W_{st} \). In earlier work, we found that using the model for \( Y_j = \sum_{s=1}^{N} U_{is} Y_{sj} \) resulted in predictors that included shrinkage constants that depended on the average response error variance. We investigate here possible alternative ways of expressing a model that would result in shrinkage constants that make use of the response error variance for realized subjects, not the average response error variance. We would like to consider an alternative expression to the second random variable in the sum \( U_{is} \frac{Y_{st}}{M} + U_{is} U_{it}^{(t)} W_{st} \). The idea is to consider an expression for a random variable similar to \( U_{it}^{(t)} \) that is conditional on \( U_{is} \).
The random variable $U_{is}$ can take on two values, zero and one. For this reason, considering random variables similar to $U_j^{(s)}$ that are conditional on $U_{is}$ results in two random variables, with one random variable corresponding to the condition that $U_{is} = 0$, and the other random variable corresponding to the condition that $U_{is} = 1$. We introduce new notation to represent these random variables.

Initially, we may consider representing the two random variables by $U_j^{(u_s)}$, where $u_s$ has a value of zero or one. This notation suffers from the problem that if zero (or one) is substituted in for $u_s$, the random variable apparently loses its connection to $i$ and $s$. For this reason, we do not consider this notation further.

An alternative possible notation is $U_j^{(s|0)}$ and $U_j^{(s|1)}$. This notation is somewhat cumbersome due to the three superscripts, but has the advantage of being clearly defined. An additional advantage is that two random variables are defined, clearly separating the settings where the realization of $U_{is}$ differs.

A different possible notation is to define a random variable equal to $u_{is}U_j^{(s)}$, say $U_j^{(s)}$. This idea was used in c07ed51.doc to explore a possible way of introducing conditionality into $U_j^{(s)}$.

2.2. The Joint Distribution of $U_{is}$ and $U_j^{(s)}$

The random variable $U_j^{(s)}$ is un-conditional. Each of the possible alternative random variables is in some sense conditional. We first consider the joint distribution of $U_j^{(s)}$ and $U_{is}$ prior to examining possible conditional distributions. The joint distribution of $U_j^{(s)}$ and $U_{is}$ is given in Table 1.

<table>
<thead>
<tr>
<th>$P(U_{is}U_j^{(s)} = u_{is}U_j^{(s)})$ (times $\frac{1}{NM}$)</th>
<th>$u_j^{(s)}$</th>
<th>$P(U_{is} = u_{is})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$M$</td>
</tr>
<tr>
<td>0</td>
<td>$N - 1$</td>
<td>$M(1 - M)$</td>
</tr>
</tbody>
</table>

Notice that expanding the expression for $U_j^{(s)}$, we can write

$$P(U_j^{(s)} = 1) = P(U_{is} = 1)P(U_j^{(s)} = 1|U_{is} = 1) + P(U_{is} = 0)P(U_j^{(s)} = 1|U_{is} = 0).$$

Using the data in Table 1, this is given by

$$P(U_j^{(s)} = 1) = \left(\frac{1}{N}\right)\left(\frac{1}{M}\right) + \left(\frac{N-1}{N}\right)\left(\frac{1}{M}\right) = \frac{1}{M}. $$

Table 1. Example of Joint Distribution of $U_j^{(s)}$ and $U_{is}$ where $M = 10$ and $N = 100$
The elementary random variables in our model are given by
\[ U_{is} \frac{Y_s}{M} + U_{ij} U^{(s)}_{ji} W_{si}. \]

Using Table 1, let us consider the expected value and variance of \( U_{ij} U^{(s)}_{ji} \). This can be evaluated simply by expressing the possible outcomes and their respective probabilities, as in Table 1a.

Table 1a. Possible Outcomes and Probabilities of \( U_{ij} U^{(s)}_{ji} \)

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{NM-1}{NM} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{NM} )</td>
</tr>
</tbody>
</table>

Using Table 1a, \( E_{\xi_1 \xi_2} \left( U_{is} U^{(s)}_{ji} \right) = \frac{1}{NM} \) and \( \text{var}_{\xi_1 \xi_2} \left( U_{is} U^{(s)}_{ji} \right) = \frac{1}{NM} \left( 1 - \frac{1}{NM} \right) \).

The variance of a product of random variables can be obtained by the conditional variance expansion,

\[ \text{var}_{\xi_1 \xi_2} (XZ) = E_{\xi_1} \left[ \text{var}_{\xi_2 \xi_1} (XZ) \right] + \text{var}_{\xi_1 \xi_2} \left[ E_{\xi_1} (XZ) \right] \]

where \( \xi_1 \) indicates expectation with respect to \( X \), and \( \xi_2 \) indicates expectation with respect to \( Z \). For this setting,

\[ \text{var}_{\xi_1 \xi_2} \left( U_{is} U^{(s)}_{ji} \right) = E_{\xi_1} \left[ \text{var}_{\xi_2 \xi_1} \left( U_{is} U^{(s)}_{ji} \right) \right] + \text{var}_{\xi_1 \xi_2} \left[ E_{\xi_1} \left( U_{is} U^{(s)}_{ji} \right) \right] \]

\[ = E_{\xi_1} \left[ U_{is} \text{var}_{\xi_2 \xi_1} \left( U^{(s)}_{ji} \right) | U_{is} = u_{is} \right] + \text{var}_{\xi_1 \xi_2} \left[ U_{is} E_{\xi_1 \xi_2} \left( U^{(s)}_{ji} \right) | U_{is} = u_{is} \right] \]

Using Table 1, \( \text{var}_{\xi_1 \xi_2} \left( U^{(s)}_{ji} | U_{is} = u_{is} \right) = \frac{1}{M} \left( 1 - \frac{1}{M} \right) \) when \( u_{is} = 0 \) and when \( u_{is} = 1 \), and similarly, \( E_{\xi_1 \xi_2} \left( U^{(s)}_{ji} | U_{is} = u_{is} \right) = \frac{1}{M} \) in either case. As a result,

\[ \text{var}_{\xi_1 \xi_2} \left( U_{is} U^{(s)}_{ji} \right) = \frac{1}{M} \left( 1 - \frac{1}{M} \right) E_{\xi_1} [U_{is}] + \frac{1}{M^2} \text{var}_{\xi_1 \xi_2} [U_{is}] \]

Since \( E_{\xi_1} [U_{is}] = \frac{1}{N} \) and \( \text{var}_{\xi_1 \xi_2} [U_{is}] = \frac{1}{N} \left( 1 - \frac{1}{N} \right) \),

\[ \text{var}_{\xi_1 \xi_2} \left( U_{is} U^{(s)}_{ji} \right) = \frac{1}{M} \left( 1 - \frac{1}{M} \right) \frac{1}{N} + \frac{1}{M^2} \frac{1}{N} \left( 1 - \frac{1}{N} \right) \]

\[ = \frac{1}{NM} \left[ \left( 1 - \frac{1}{M} \right) + \frac{1}{M} \left( 1 - \frac{1}{N} \right) \right] \]

\[ = \frac{1}{NM} \left( 1 - \frac{1}{NM} \right) \]
2.3. **Proposal 1: The Joint Distribution of** $U_{jt}^{(i\bar{s})}$ and $U_{jt}^{(s\bar{i})}$ and $U_{is}$.

We may develop this later.

2.4. **Proposal 2: The Joint Distribution of** $U_{jt}^{(i\bar{s})} = u_{is}U_{jt}^{(s\bar{i})}$ and $U_{is}$.

Consider a random variable $U_{jt}^{(i\bar{s})} = u_{is}U_{jt}^{(s\bar{i})}$, and the joint distribution of $U_{jt}^{(i\bar{s})}$ and $U_{is}$, as given in Table 2.

<table>
<thead>
<tr>
<th>$P(U_{is}U_{jt}^{(i\bar{s})} = u_{is}U_{jt}^{(s\bar{i})})$ (times $\frac{1}{NM}$)</th>
<th>$u_{is}^{(i\bar{s})}$</th>
<th>$P(U_{is} = u_{is})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_{is}$</td>
<td>1</td>
<td>$M - 1$</td>
</tr>
<tr>
<td>0</td>
<td>$N - 1$</td>
<td>$M - 1$</td>
</tr>
</tbody>
</table>

Notice the difference in Table 2 between the probabilities associated with the outcomes for $u_{is}^{(i\bar{s})} = 1, u_{is} = 0$ as compared with Table 1.

The elementary random variables that we have been considering are given by $U_{is} = \frac{Y_{s}}{M} + U_{is}U_{jt}^{(s\bar{i})}$. We propose using a different elementary random variables defined by $U_{is} = \frac{Y_{s}}{M} + U_{is}U_{jt}^{(i\bar{s})}$.

Using Table 2, let us consider the expected value and variance of $U_{is}U_{jt}^{(i\bar{s})}$. This can be evaluated simply by expressing the possible outcomes and their respective probabilities, as in Table 2a.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{NM - 1}{NM}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{NM}$</td>
</tr>
</tbody>
</table>

Using Table 2a, $E_{\tilde{t}\tilde{s}}(U_{is}U_{jt}^{(i\bar{s})}) = \frac{1}{NM}$ and $\text{var}_{\tilde{t}\tilde{s}}(U_{is}U_{jt}^{(i\bar{s})}) = \frac{1}{NM} \left(1 - \frac{1}{NM}\right)$. This expression is identical to variance of $U_{is}U_{jt}^{(s\bar{i})}$, even though the joint distribution of $U_{is}$ and $U_{jt}^{(i\bar{s})}$ is different from the joint distribution of $U_{is}$ and $U_{jt}^{(s\bar{i})}$.
We can evaluate the variance of \( U_{is} U_{jt}^{(is)} \) by the conditional variance expansion given by

\[
\text{var}_{\xi_{i|s}}(U_{is} U_{jt}^{(is)}) = E_{\xi_i} \left[ \text{var}_{\xi_{j|t}}(U_{is} U_{jt}^{(is)}) \right] + \text{var}_{\xi_i} \left[ E_{\xi_{j|t}} (U_{is} U_{jt}^{(is)}) \right]
\]

\[
= E_{\xi_i} \left[ U_{is} \text{var}_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = u_{is}) \right] + \text{var}_{\xi_i} \left[ U_{is} E_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = u_{is}) \right].
\]

Since \( U_{jt}^{(is)} = u_{jt} U_{jt}^{(is)} \), we need to consider the two possible outcomes for \( \text{var}_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = u_{is}) \) or \( E_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = u_{is}) \) when evaluating these terms. First, using Table 2,

\[
U_{is} E_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = 0) = 0 \text{ while } U_{is} \text{var}_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = 0) = 0,
\]

and

\[
U_{is} E_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = 1) = \frac{1}{M} U_{is} \text{ while } U_{is} \text{var}_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = 1) = \frac{1}{M} \left(1 - \frac{1}{M}\right).
\]

To evaluate the expression for \( \text{var}_{\xi_i}[E_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] \), we use

\[
\text{var}_{\xi_i}[E_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] = \sum_{u_{is}=0}^{1} \left( U_{is} E_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = u_{is}) - E_{\xi_i}[E_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] \right)^2 P(U_{is} = u_{is})
\]

where

\[
E_{\xi_i}[E_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] = \sum_{u_{is}=0}^{1} U_{is} E_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = u_{is}) P(U_{is} = u_{is}).
\]

Now

\[
E_{\xi_i}[E_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] = \frac{1}{M} P(U_{is} = 1) = \frac{1}{NM}.
\]

Then

\[
\text{var}_{\xi_i}[E_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] = \left( \frac{1}{NM} \right)^2 \frac{N-1}{N} + \left( \frac{1}{M} - \frac{1}{NM} \right)^2 \left( \frac{1}{N} \right)
\]

\[
= \left( \frac{1}{NM} \right)^2 \left[ \frac{N-1}{N} + \frac{N-1}{N} \right]
\]

\[
= (N-1) \left( \frac{1}{NM} \right)^2
\]

Next, let us evaluate \( E_{\xi_i}[\text{var}_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] = E_{\xi_i}[U_{is} \text{var}_{\xi_{j|t}}(U_{jt}^{(is)} | U_{is} = u_{is})] \). This is given by

\[
E_{\xi_i}[\text{var}_{\xi_{j|t}}(U_{is} U_{jt}^{(is)})] = 0(0) P(U_{is} = 0) + \frac{1}{M} \left(1 - \frac{1}{M}\right) P(U_{is} = 1)
\]

\[
= \frac{1}{NM} \left(1 - \frac{1}{M}\right).
\]

Combining these terms,
\[ \text{var}_{\xi_{ij}} \left( U_{ij} U_{ji}^{(iz)} \right) = E_{\xi_{ij}} \left[ \text{var}_{\xi_{ij}} \left( U_{ij} U_{ji}^{(iz)} \right) \right] + \text{var}_{\xi_{ij}} \left( E_{\xi_{ij}} \left( U_{ij} U_{ji}^{(iz)} \right) \right) \]
\[ = \frac{1}{NM} \left( 1 - \frac{1}{M} \right) + (N - 1) \left( \frac{1}{NM} \right)^2 \]
\[ = \frac{1}{NM} \left[ 1 - \frac{1}{M} + \frac{1}{M} - \frac{1}{N} \right] \]
\[ = \frac{1}{NM} \left[ 1 - \frac{1}{NM} \right] \]