Simpler General Expression for the EMSE for the Partially Expanded Predictor

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Introduction

The general expression for the EMSE developed given in c07ed16.doc simplifies to a formula for the PSU sample mean that matches the simulation results given in c07ed09.doc. In this document, we express the general EMSE using simpler notation. The expression used for the general EMSE is given by

\[
\text{var}_{\hat{T}} (\hat{T} - T) = c_f \left[ \left( \sigma_{r_1}^2 + \bar{q}_1 \right) \mathbf{I}_n \right] + \left( \bar{r}_1^2 - \frac{1}{N} \sigma_{r_1}^2 \right) \mathbf{J}_n \]  

\[
- 2c_f \left[ \left( \sigma_{r_{12}} + \bar{q}_2 \right) \mathbf{I}_n \right] + \left( \bar{r}_{12}^2 - \frac{1}{N} \sigma_{r_{12}} \right) \mathbf{J}_n \]  

\[
+ \mathbf{1}_n \left[ \left( \sigma_{r_2}^2 + \bar{q}_3 \right) \mathbf{I}_n \right] + \left( \bar{r}_2^2 - \frac{1}{N} \sigma_{r_2}^2 \right) \mathbf{J}_n \]  

\[
+ 2\mathbf{1}_n \left[ \sigma_{r_{23}} \left( \mathbf{I}_n + \frac{1}{n} \mathbf{0} \right) \right] + \left( \bar{r}_{23}^2 - \frac{1}{N} \sigma_{r_{23}} \right) \mathbf{J}_n \right] c
\]

\[
- 2c_f \left[ \sigma_{r_{13}} \left( \mathbf{I}_n + \frac{1}{n} \mathbf{0} \right) \right] + \left( \bar{r}_{13}^2 - \frac{1}{N} \sigma_{r_{13}} \right) \mathbf{I}_n^T \mathbf{c} + \mathbf{c}^T \left[ \left( \sigma_{r_3}^2 \right) \mathbf{I}_n + \left( \bar{r}_3^2 - \frac{1}{N} \sigma_{r_3} \right) \mathbf{J}_n \right] \mathbf{c}
\]

\[
- \frac{1}{N} \sum_{s=1}^{S} (n \bar{c}_s a^*_s - n b_s - N \bar{c}_s M_s w_s) \mu_s^2
\]

First, note that \( r_{s_1} = a^*_s \mu_s \), \( r_{s_2} = b_s \mu_s \), \( r_{s_3} = M_s w_s \mu_s \) where \( a^*_s = k^*_s M_s w_s \), \( b_s = \left( \bar{c}_s \left( k^*_s - 1 \right) \right) M_s w_s \). We use these definitions to express these terms in a different form.

Alternative Expressions for \( r_{s_3} = M_s w_s \mu_s \).

We define \( r_s = r_{s_3} \). As a result, \( r_s = M_s w_s \mu_s \).

Alternative Expressions for \( r_{s_1} = a^*_s \mu_s \)

We first use \( a^*_s = k^*_s M_s w_s \) to express \( r_{s_1} = k^*_s \left( M_s w_s \mu_s \right) \). Then, noting that \( r_s = M_s w_s \mu_s \), we express \( r_{s_1} = k^* r_s \).

Alternative Expressions for \( r_{s_2} = b_s \mu_s \)

Notice that we can express \( b_s = \left( \bar{c}_s \left( k^*_s - 1 \right) \right) M_s w_s \). As a result,

\[
r_{s_2} = \left( \bar{c}_s \left( k^*_s - 1 \right) \right) M_s w_s \mu_s .
\]

Using \( r_s = M_s w_s \mu_s \), \( r_{s_2} = \left( \bar{c}_s \left( k^*_s - 1 \right) \right) r_s \). Let us define \( \alpha_s = \bar{c}_s \left( k^*_s - 1 \right) \). Then \( r_{s_2} = \alpha_s r_s \).

Alternative Expressions for \( \bar{r}_3 \) and \( \sigma_{r_3}^2 \).
Recall that we define \( r_s = r_{3s} \). We represent \( r_3 \) by \( \mu_r = \frac{1}{N} \sum_{s=1}^{N} r_s \), and

\[
\sigma_r^2 = \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)^2.
\]

As a result, the equivalent notation for \( \bar{r}_3 \) and \( \sigma_r^2 \) is \( \mu_r \) and \( \sigma_r^2 \).

**Alternative Expressions for \( \bar{r}_1 \) and \( \sigma_{r_1}^2 \)**

Recall that we define \( r_{1s} = k_s r_s \). We represent \( r_1 \) by \( \mu_{kr} = \frac{1}{N} \sum_{s=1}^{N} k_s r_s \), and

\[
\sigma_{kr}^2 = \frac{1}{N-1} \sum_{s=1}^{N} (k_s r_s - \mu_{kr})^2.
\]

As a result, the equivalent notation for \( \bar{r}_1 \) and \( \sigma_{r_1}^2 \) is \( \mu_{kr} \) and \( \sigma_{kr}^2 \).

**Alternative Expressions for \( \bar{r}_2 \) and \( \sigma_{r_2}^2 \)**

Recall that we define \( r_{2s} = \alpha_s r_s \). We represent \( r_2 \) by \( \mu_{ar} = \frac{1}{N} \sum_{s=1}^{N} \alpha_s r_s \), and

\[
\sigma_{ar}^2 = \frac{1}{N-1} \sum_{s=1}^{N} (\alpha_s r_s - \mu_{ar})^2.
\]

As a result, the equivalent notation for \( \bar{r}_2 \) and \( \sigma_{r_2}^2 \) is \( \mu_{ar} \) and \( \sigma_{ar}^2 \).

**Alternative Expressions for \( \sigma_{r_{12}}, \sigma_{r_{13}}, \text{ and } \sigma_{r_{23}} \)**

Recall that we define \( r_{1s} = k_s r_s, r_{2s} = \alpha_s r_s \), and \( r_{3s} = r_s \). Then we define

\[
\sigma_{r_{12}} = \sigma_{rk,ra} = \frac{1}{N-1} \sum_{s=1}^{N} (k_s r_s - \mu_{kr})(\alpha_s r_s - \mu_{ar}),
\]

\[
\sigma_{r_{13}} = \sigma_{rk,r} = \frac{1}{N-1} \sum_{s=1}^{N} (k_s r_s - \mu_{kr})(r_s - \mu_r),
\]

\[
\sigma_{r_{23}} = \sigma_{r,ra} = \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)(\alpha_s r_s - \mu_{ar}).
\]

As a result, the equivalent notation for \( \sigma_{r_{12}}, \sigma_{r_{13}} \) and \( \sigma_{r_{23}} \) is \( \sigma_{rk,ra}, \sigma_{rk,r} \), and \( \sigma_{r,ra} \).

**Alternative Expressions for \( q_{1s}, q_{2s}, \text{ and } q_{3s}. \)**
Recall that we define $q_{1s} = a_s^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right)$, $q_{2s} = a_s^* b_s \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right)$, and 

$q_{3s} = b_s^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right)$ where $a_s^* = k_s^* M_s w_s$ and $b_s = \alpha_s M_s w_s$ where $\alpha_s = \overline{c}_i \left( k_s^* - 1 \right) - \frac{N-n}{n} \overline{c}_i$.

Then $q_{1s} = k_s^2 (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right)$, $q_{2s} = \alpha_s k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right)$, and 

$q_{3s} = \alpha_s^2 (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right)$. As a result, the equivalent notation for $q_{1s}$, $q_{2s}$ and $q_{3s}$ is

$$q_{1s} = k_s^2 (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right), \quad q_{2s} = \alpha_s k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right) \text{ and}$$

$$q_{3s} = \alpha_s^2 (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_i}{M_s} \right).$$

Replacing Expression for EMSE with Alternative Notation

We use the alternative notation to express the EMSE. It is given by

$$\text{var}_{\hat{\beta}_s} (\hat{T} - T) = c'_i \left[ \left( \sigma^2 + \overline{q}_1 \right) I_n + \left( \frac{\mu^2}{N} - \sigma^2 \right) J_n \right] c_i - 2c'_i \left[ \left( \sigma_{r,r} + \overline{q}_2 \right) I_n + \left( \mu_{s,s} - \frac{1}{N} \sigma_{r,r} \right) J_n \right] c_i$$

$$+ 2c'_i \left[ \left( \sigma_{a,a} + \overline{q}_3 \right) I_n + \left( \mu_{a,a} - \frac{1}{N} \sigma_{a,a} \right) J_n \right] c_i + 2\overline{c}'_i \left[ \left( \mu_{s,s} - \frac{1}{N} \sigma_{r,r} \right) I_n \right] c_i + c'_i \left[ \sigma^2 I_n + \left( \mu_r^2 - \frac{1}{N} \sigma_r^2 \right) J_n \right] c_i$$

$$- 2c'_i \left[ \sigma^2 I_n \right. \left. \sum_{n=1}^N \left( n\overline{c}_i - n\overline{c}_j - N\overline{c}_i \right) \right] c_i$$

We make use of the expressions, $c'_i I_n c_j = \sum_{i=1}^n c_i^2$, $c'_i J_n c_j = n^2 \overline{c}_i^2$, $c'_i I_n 1_n = n \overline{c}_i$, $c'_i J_n 1_n = n^2 \overline{c}_i$, $1'_n 1_n 1_n = n$, $1'_n J_n 1_n = n^2$, $1'_n \left[ I_n \mid 0 \right] \left[ n(N-n) \right] c_i = n\overline{c}_i$, $1'_n I_n' 1_n c_i = nN\overline{c}_i$, $c'_i 1_n' I_n 1_n c_i = n\overline{c}_i$, $1'_n J_n c_i = \sum_{i=1}^n c_i^2$, and $c'_i J_n c_i = N^2 \overline{c}_i^2$ to simplify these expressions. Since

$$\sum_{i=1}^n (c_i - \overline{c}_i)^2 = \sum_{i=1}^n c_i^2 - n\overline{c}_i^2, \quad \sum_{i=1}^n c_i^2 = n\overline{c}_i^2 + \sum_{i=1}^n (c_i - \overline{c}_i)^2$$

and hence $c'_i I_n c_i = n\overline{c}_i^2 + \sum_{i=1}^n (c_i - \overline{c}_i)^2$. 

while \( \mathbf{c}'_i \left( \mathbf{I}_n \mid \mathbf{0}_{n(N-n)} \right) \mathbf{c} = n \bar{c}_i^2 + \sum_{i=1}^{n} (c_i - \bar{c})^2 \). In addition, \( \sum_{i=1}^{N} c_i^2 = N \bar{c}^2 + \sum_{i=1}^{N} (c_i - \bar{c})^2 \) so that

\[
\mathbf{c}' \mathbf{I}_N \mathbf{c} = N \bar{c}^2 + \sum_{i=1}^{N} (c_i - \bar{c})^2.
\]

Using these expressions,

\[
\begin{align*}
\text{var}_{\small \hat{T}} (\hat{T} - T) &= (\sigma^2_v + \bar{q}_i^2) \left( n \bar{c}_i^2 + \sum_{i=1}^{n} (c_i - \bar{c})^2 \right) + \left( \mu_r - \frac{1}{N} \sigma^2_v \right) n^2 \bar{c}_i^2 \\
&- 2(\sigma_{rk,ra} + \bar{q}_i) n \bar{c}_i - 2 \left( \mu_r \mu_{ar} - \frac{1}{N} \sigma^2_v \right) n \bar{c}_i \\
&+ \left( \sigma^2_{ar} + \bar{q}_i \right) n + \left( \mu^2_{ar} - \frac{1}{N} \sigma^2_{ar} \right) n \\
&+ 2 \sigma_{r,ra} n \bar{c}_i + 2 \left( \mu_{ar} \mu_r - \frac{1}{N} \sigma^2_{r,ra} \right) n N \bar{c}_i \\
&- 2 \sigma^2_{rk} \left( n \bar{c}_i^2 + \sum_{i=1}^{n} (c_i - \bar{c})^2 \right) - 2 \left( \mu_r \mu_{ar} - \frac{1}{N} \sigma^2_{rk,ra} \right) n \bar{c}_i N \bar{c}_i \\
&+ \sigma^2_{r} \left( N \bar{c}^2 + \sum_{i=1}^{N} (c_i - \bar{c})^2 \right) + \left( \mu_r - \frac{1}{N} \sigma^2_r \right) N^2 \bar{c}^2 \\
&- \left( \frac{1}{N} \sum_{i=1}^{N} \left( n \bar{c}_i k_i - n \alpha_s - N \bar{c} \right) r \right)^2
\end{align*}
\]

Grouping terms,

\[
\begin{align*}
\text{var}_{\small \hat{T}} (\hat{T} - T) &= \\
&\sigma^2_v \left( n \bar{c}_i^2 + \sum_{i=1}^{n} (c_i - \bar{c})^2 \right) + \bar{q}_i \left( n \bar{c}_i^2 + \sum_{i=1}^{n} (c_i - \bar{c})^2 \right) + \mu_r^2 n^2 \bar{c}_i^2 - \frac{1}{N} \sigma^2_v n^2 \bar{c}_i^2 \\
&- 2 \sigma_{rk,ra} n \bar{c}_i - 2 \bar{q}_i n \bar{c}_i - 2 \mu_r \mu_{ar} n^2 \bar{c}_i + 2 \frac{1}{N} \sigma_{rk,ra} n^2 \bar{c}_i \\
&+ n \sigma^2_{ar} + n \bar{q}_i + n^2 \mu_{ar}^2 - \frac{n^2}{N} \sigma^2_{ar} \\
&+ 2 \sigma_{r,ra} n \bar{c}_i + 2 \mu_{ar} \mu_r n N \bar{c} - 2 \frac{1}{N} \sigma_{r,ra} n N \bar{c} \\
&- 2 \sigma^2_{rk} n \bar{c}_i^2 - 2 \sigma_{rk,ra} \sum_{i=1}^{n} (c_i - \bar{c})^2 + 2 \mu_{rk} \mu_{ra} n \bar{c}_i N \bar{c} + 2 \frac{1}{N} \sigma_{rk,ra} n \bar{c}_i N \bar{c} \\
&+ N \bar{c}^2 \sigma^2_v + \sigma^2_r \sum_{i=1}^{N} (c_i - \bar{c})^2 + N^2 \bar{c}^2 \mu_r^2 - \frac{1}{N} N^2 \bar{c}^2 \sigma^2_r \\
&- \left( \frac{1}{N} \sum_{i=1}^{N} \left( n \bar{c}_i k_i - n \alpha_s - N \bar{c} \right) r \right)^2
\end{align*}
\]

or
\[ \text{var}_{\xi \xi}(\hat{T} - T) = \]
\[ \sigma_{\nu}^2 \left[ n\overline{c}_i^2 + \sum_{i=1}^{n} (c_i - \overline{c}_i)^2 \right] - \frac{1}{N} \sigma_{\nu}^2 n^2 \overline{c}_i^2 \]
\[ + \bar{q}_i \left[ n\overline{c}_i^2 + \sum_{i=1}^{n} (c_i - \overline{c}_i)^2 \right] - 2\bar{q}_3 n\overline{c}_i + n\bar{q}_3 \]
\[ + \mu_{\nu}^2 n^2 \overline{c}_i^2 - 2\mu_{\nu}\mu_{ar} n^2 \overline{c}_i + n^2 \mu_{ar}^2 + 2\mu_{ar}\mu_{nN}\overline{c}_i - 2\mu_{\nu}\mu_{nN}\overline{c}_i \overline{c}_i + N^2 \overline{c}_i^2 \mu_{r}^2 \]
\[ - 2\sigma_{\nu,ra} n\overline{c}_i + 2 \frac{1}{N} \sigma_{\nu,ra} n^2 \overline{c}_i \]
\[ + n\sigma_{ar}^2 \frac{n^2}{N} \sigma_{ar}^2 \]
\[ + 2\sigma_{\nu,ar} n\overline{c}_i - 2 \frac{1}{N} \sigma_{\nu,ra} n N\overline{c}_i \]
\[ - 2\sigma_{\nu,ra} n\overline{c}_i - 2\sigma_{\nu,ra} \sum_{i=1}^{n} (c_i - \overline{c}_i)^2 + 2 \frac{1}{N} \sigma_{\nu,ra} n\overline{c}_i N\overline{c}_i \]
\[ + N\overline{c}_i^2 \sigma_{r}^2 + \sigma_{r}^2 \sum_{i=1}^{n} (c_i - \overline{c}_i)^2 - \frac{1}{N} N^2 \overline{c}_i^2 \sigma_{r}^2 \]
\[ - \left( \frac{1}{N} \sum_{i=1}^{N} (n\overline{c}_i k_i^* - n\alpha_s - N\overline{c}_i) r_i \right)^2 \]

or

\[ \text{var}_{\xi \xi}(\hat{T} - T) = \]
\[ \sigma_{\nu}^2 \left[ n\overline{c}_i^2 \left( 1 - \frac{n}{N} \right) + \sum_{i=1}^{n} (c_i - \overline{c}_i)^2 \right] \]
\[ + \bar{q}_i \left[ n\overline{c}_i^2 + \sum_{i=1}^{n} (c_i - \overline{c}_i)^2 \right] - 2\bar{q}_3 n\overline{c}_i + n\bar{q}_3 \]
\[ + n^2 \left( \mu_{\nu}^2 \overline{c}_i^2 - 2\mu_{\nu}\mu_{ar} \overline{c}_i + \mu_{ar}^2 \right) + N\overline{c}_i \mu_r \left( 2\mu_{ar}n - 2\mu_{ar}n\overline{c}_i + N\overline{c}_i \mu_r \right) \]
\[ - 2n\overline{c}_i \sigma_{\nu,ra} \left( 1 - \frac{n}{N} \right) \]
\[ + n\sigma_{ar}^2 \left( 1 - \frac{n}{N} \right) \]
\[ + 2n\sigma_{\nu,ra} (\overline{c}_i - \overline{c}) \]
\[ - 2\sigma_{\nu,ra} \left[ n\overline{c}_i (\overline{c}_i - \overline{c}) + \sum_{i=1}^{n} (c_i - \overline{c}_i)^2 \right] \]
\[ + \sigma_{r}^2 \sum_{i=1}^{n} (c_i - \overline{c})^2 \]
\[ - \left( \frac{1}{N} \sum_{i=1}^{N} (n\overline{c}_i k_i^* - n\alpha_s - N\overline{c}_i) r_i \right)^2 \]

We express this as
\[
\text{var}_{\hat{\nu}} \left( \hat{T} - T \right) = n \left( 1 - \frac{n}{N} \right) \left( \sigma_{\nu}^2 \bar{c}_i^2 - 2 \bar{c}_i \sigma_{\nu,ra} + \sigma_{ar}^2 \right) \\
+ \left( \sigma_{\nu}^2 - 2 \sigma_{\nu,ra} + \bar{q}_i \right) \left( \sum_{i=1}^{n} \left( c_i - \bar{c}_i \right)^2 \right) + n \left( \bar{c}_i \bar{q}_i - 2 \bar{q}_i \bar{c}_i + \bar{q}_i \right) \\
+ n^2 \left( \bar{c}_i \mu_{\nu} - \mu_{ar} \right)^2 - 2 N \bar{c}_i \mu_{\nu} n \left( \bar{c}_i \mu_{\nu} - \mu_{ar} \right) + \left( N \bar{c}_i \mu_{\nu} \right)^2 \\
+ 2n \left( \bar{c}_i - \bar{c} \right) \left( \sigma_{r,ra} - \bar{c}_i \sigma_{r,ra} \right) + \sigma_r^2 \left( \sum_{i=1}^{N} \left( c_i - \bar{c} \right)^2 \right) - \left( \frac{1}{N} \sum_{i=1}^{N} \left( n \bar{c}_i k_s^* - n \alpha_s - N \bar{c}_i \right) r_s \right)^2
\]

which reduces to
\[
\text{var}_{\hat{\nu}} \left( \hat{T} - T \right) = n \left( 1 - \frac{n}{N} \right) \left( \sigma_{\nu}^2 \bar{c}_i^2 - 2 \bar{c}_i \sigma_{\nu,ra} + \sigma_{ar}^2 \right) + \left( \sigma_{\nu}^2 - 2 \sigma_{\nu,ra} + \bar{q}_i \right) \left( \sum_{i=1}^{n} \left( c_i - \bar{c}_i \right)^2 \right) \\
+ n \left( \bar{c}_i \bar{q}_i - 2 \bar{q}_i \bar{c}_i + \bar{q}_i \right) + \left[ n \left( \bar{c}_i \mu_{\nu} - \mu_{ar} \right) - N \bar{c}_i \mu_{\nu} \right]^2 + 2n \left( \bar{c}_i - \bar{c} \right) \left( \sigma_{r,ra} - \bar{c}_i \sigma_{r,ra} \right) \\
+ \sigma_r^2 \left( \sum_{i=1}^{N} \left( c_i - \bar{c} \right)^2 \right) - \left( \frac{1}{N} \sum_{i=1}^{N} \left( n \bar{c}_i k_s^* - n \alpha_s - N \bar{c}_i \right) r_s \right)^2
\]

Terms in this expression are defined by \( \sigma_{\nu}^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left( k_s^* r_s - \mu_{ar} \right)^2 \),
\( \sigma_{\nu,ra} = \frac{1}{N-1} \sum_{i=1}^{N} \left( k_s^* r_s - \mu_{ar} \right) \left( \alpha_s r_s - \mu_{ar} \right) \), \( \sigma_{ar}^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left( \alpha_s r_s - \mu_{ar} \right)^2 \), \( \sigma_r^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left( r_s - \mu_r \right)^2 \),
\( \sigma_{r,ra} = \frac{1}{N-1} \sum_{i=1}^{N} \left( r_s - \mu_r \right) \left( \alpha_s r_s - \mu_{ar} \right) \) where

\[
r_s = M_s w_s \mu_{\nu}, \quad \alpha_s = \bar{c}_i \left( k_s^* - 1 \right) - \frac{N-n}{n} \bar{c}_i, \quad k_s = k_s - \frac{k_s}{M_s w_s \mu_{\nu}} \sum_{i=1}^{N} \left( 1 - k_s \right) \left( M_i w_i \mu_{\nu} \right) \text{ and}
\]

\[
k_s = \frac{\mu_s^2}{\mu_s^2 + \left( 1 - f_s \right) \frac{N-1}{N} \sigma_s^2 m_s}.
\]

Also, \( \mu_r = \frac{1}{N} \sum_{i=1}^{N} r_s \), \( \mu_{ar} = \frac{1}{N} \sum_{i=1}^{N} k_s^* r_s \), and \( \alpha_r = \frac{1}{N} \sum_{i=1}^{N} \alpha_s r_s \), and

\[
q_{1s} = k_s^* \left( M_s w_s \right)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right), \quad q_{2s} = \alpha_s k_s^* \left( M_s w_s \right)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \text{ and}
\]

\[
q_{3s} = \alpha_s^2 \left( M_s w_s \right)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \text{ where } \bar{q}_1 = \frac{1}{N} \sum_{i=1}^{N} q_{1s}, \quad \bar{q}_2 = \frac{1}{N} \sum_{i=1}^{N} q_{2s}, \text{ and } \bar{q}_3 = \frac{1}{N} \sum_{i=1}^{N} q_{3s}. \text{ Finally,}
\]

\[
\bar{c}_i = \frac{1}{n} \sum_{i=1}^{n} c_i \text{ and } \bar{c} = \frac{1}{N} \sum_{i=1}^{N} c_i.
\]

Special Case when Predicting a PSU Mean.
Some simplifications occur when predicting a PSU mean. In this case $w_i = \frac{1}{M_s}$. Then

$$r_i = \mu_i, \quad \mu_r = \mu, \quad \sigma_r^2 = \sigma^2 = \frac{1}{N-1} \sum_{s=1}^{N} (\mu_s - \mu)^2, \quad \bar{c} = \frac{1}{N}, \quad \sum_{i=1}^{N} (c_i - \bar{c})^2 = 1 - \frac{1}{N},$$

$$\sum_{i=1}^{N} (c_i - \bar{c})^2 = 1 - \frac{1}{N}$$

and $n\bar{c}_i + (N-n)\bar{c}_r = 1$. Using these expressions,

$$\text{var}_{\bar{c}_s} \left( \tilde{T} - T \right) = n \left( 1 - \frac{n}{N} \right) \left( \sigma_r^2 \bar{c}_i^2 - 2 \bar{c}_i \sigma_{r,a} + \sigma_{a,r}^2 \right) + \left( \sigma_r^2 - 2 \sigma_{r,a} + \bar{q}_1 \right) \left( 1 - \frac{1}{n} \right)$$

$$+ n \left( \bar{c}_i^2 \bar{q}_1 - 2 \bar{q}_2 \bar{c}_i + \bar{q}_3 \right) + n \left( \bar{c}_i \mu_r - \mu_{ar} \right) - \mu \right)^2 + 2n \left( \bar{c}_i - \frac{1}{N} \right) \left( \sigma_{r,a} - \bar{c}_i \sigma_{r,r} \right) \cdot$$

$$+ \sigma_r^2 \left( 1 - \frac{1}{N} \right) - \left( \frac{1}{N} \sum_{s=1}^{N} \left( n\bar{c}_i k_s^* - n\alpha_s - 1 \right) \mu_s \right)^2$$

Now $\alpha_s = \bar{c}_i \left( k_s^* - 1 \right) - \frac{N-n}{n} \bar{c}_r \mu_s$, so that

$$n\bar{c}_i k_s^* - n\alpha_s - 1 = n\bar{c}_i k_s^* - n \left( \bar{c}_i \left( k_s^* - 1 \right) - \frac{N-n}{n} \bar{c}_r \mu_s \right) - 1$$

$$= n\bar{c}_i k_s^* - n\bar{c}_i \left( k_s^* - 1 \right) + (N-n)\bar{c}_r \mu_s - 1$$

$$= n\bar{c}_i + (N-n)\bar{c}_r \mu_s - 1$$

$$= N\bar{c} - 1$$

$$= 0$$

Then,

$$\text{var}_{\bar{c}_s} \left( \tilde{T} - T \right) = n \left( 1 - \frac{n}{N} \right) \left( \sigma_r^2 \bar{c}_i^2 - 2 \bar{c}_i \sigma_{r,a} + \sigma_{a,r}^2 \right) + \left( \sigma_r^2 - 2 \sigma_{r,a} + \bar{q}_1 \right) \left( 1 - \frac{1}{n} \right)$$

$$+ n \left( \bar{c}_i^2 \bar{q}_1 - 2 \bar{q}_2 \bar{c}_i + \bar{q}_3 \right) + n \left( \bar{c}_i \mu_r - \mu_{ar} \right) - \mu \right)^2 + 2n \left( \bar{c}_i - \frac{1}{N} \right) \left( \sigma_{r,a} - \bar{c}_i \sigma_{r,r} \right) + \sigma_r^2 \left( 1 - \frac{1}{N} \right) \cdot$$

**Simplifications when the PSU mean is in the Sample.**

If the PSU is in the sample, $\bar{c}_i = \frac{1}{n}$ and $\bar{c}_r = 0$. As a result,

$$\text{var}_{\bar{c}_s} \left( \tilde{T} - T \right) = n \left( 1 - \frac{n}{N} \right) \left( \sigma_r^2 \frac{1}{n^2} - 2 \frac{1}{n} \sigma_{r,a} + \sigma_{a,r}^2 \right) + \left( \sigma_r^2 - 2 \sigma_{r,a} + \bar{q}_1 \right) \left( 1 - \frac{1}{n} \right)$$

$$+ \left( \frac{1}{n} \bar{q}_1 - 2 \bar{q}_2 + n\bar{q}_3 \right) + \left( \mu_r - n\mu_{ar} - \mu \right)^2 + \sigma_r^2 \left( 1 - \frac{1}{N} \right) + 2n \left( \frac{1}{n} - \frac{1}{N} \right) \left( \sigma_{r,a} - \frac{1}{n} \sigma_{r,r} \right) \cdot$$
In addition, since $\alpha_s = \bar{c}_1(k^*_s - 1) - \frac{N-n}{n} \bar{c}_r$, $\alpha_s = \frac{1}{n} k^*_s - 1$. Now $\alpha_r = \frac{1}{n} k^*_r - 1$, and

$$\mu_{ar} = \frac{1}{nN} \sum_{s=1}^{N} k^*_r r_s - \frac{1}{nN} \sum_{s=1}^{N} r_s$$

or $\mu_{ar} = \frac{1}{n} \mu_{kr} - \frac{1}{n} \mu_r$. Then $\alpha_r r_s - \mu_{ar} = \frac{1}{n} (k^*_r r_s - \mu_{kr}) - \frac{1}{n} (r_s - \mu_r)$.

Using this expression,

$$\sigma_{rk,ar} = \frac{1}{N-1} \sum_{s=1}^{N} (k^*_r r_s - \mu_{kr})(\alpha_r r_s - \mu_{ar})$$

$$= \frac{1}{n} \left[ \frac{1}{N-1} \sum_{s=1}^{N} (k^*_r r_s - \mu_{kr})^2 - \frac{1}{N-1} \sum_{s=1}^{N} (k^*_r r_s - \mu_{kr})(r_s - \mu_r) \right]$$

$$= \frac{1}{n} \left( \sigma^2_{kr} - \sigma_{rk,r} \right)$$

$$\sigma^2_{ar} = \frac{1}{N-1} \sum_{s=1}^{N} (\alpha_r r_s - \mu_{ar})^2$$

$$= \frac{1}{n^2} \left[ \frac{1}{N-1} \sum_{s=1}^{N} (k^*_r r_s - \mu_{kr})^2 - \frac{1}{N-1} \sum_{s=1}^{N} (k^*_r r_s - \mu_{kr})(r_s - \mu_r) + \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)^2 \right]$$

$$= \frac{1}{n^2} \left( \sigma^2_{kr} - 2\sigma_{kr,r} + \sigma^2 \right)$$

and

$$\sigma_{r,ra} = \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)(\alpha_r r_s - \mu_{ar})$$

$$= \frac{1}{n} \left[ \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)(k^*_r r_s - \mu_{kr}) - \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)^2 \right]$$

$$= \frac{1}{n} \left( \sigma_{kr,r} - \sigma^2 \right)$$

Substituting in these expressions,

$$var_{\tilde{T}}(\tilde{T} - T) = n \left( 1 - \frac{n}{N} \right) \left( \frac{\sigma^2_{kr}}{n^2} \right) \frac{1}{n} \left( \sigma^2_{kr} - 2\sigma^2_{kr,r} + 2\sigma^2_{kr,r} - 2\sigma_{kr,r} \right)$$

$$+ \left( \sigma^2_{kr} - 2\sigma_{kr,r} + \tilde{q}_1 \right) \left( 1 - \frac{n}{N} \right) + \sigma^2 \left( 1 - \frac{n}{N} \right)$$

$$+ \left( \frac{1}{n} \tilde{q}_1 - 2\tilde{q}_2 + n\tilde{q}_3 \right) + \left( \mu_{kr} - n\mu_{ar} \right) - \mu \right)^2 + 2n \left( \frac{1}{n} - \frac{1}{N} \right) \left( \frac{1}{n} \left( \sigma_{kr} - \sigma^2 \right) - \frac{1}{n} \sigma_{kr,r} \right)$$

which simplifies to

$$var_{\tilde{T}}(\tilde{T} - T) = \frac{1}{n} \left( 1 - \frac{n}{N} \right) \left( \frac{\sigma^2_{kr}}{n^2} - 2\sigma^2_{kr} + 2\sigma_{kr,r} + \sigma^2_{kr} - 2\sigma_{kr,r} + \sigma^2 \right)$$

$$+ \left( \sigma^2_{kr} - 2\sigma_{kr,r} + \tilde{q}_1 \right) \left( 1 - \frac{n}{N} \right) + \sigma^2 \left( 1 - \frac{n}{N} \right)$$

$$+ \left( \frac{1}{n} \tilde{q}_1 - 2\tilde{q}_2 + n\tilde{q}_3 \right) + \left( \mu_{kr} - n\mu_{ar} \right) - \mu \right)^2 + 2n \left( \frac{1}{n} - \frac{1}{N} \right) \left( \sigma_{kr} - \sigma^2 - \sigma_{kr,r} \right)$$
or
\[
\text{var}_{\xi_i} \left( \hat{T} - T \right) = \left( \frac{1}{n} - \frac{1}{N} \right) \sigma^2 - 2 \left( \frac{1}{n} - \frac{1}{N} \right) \sigma^2 + \left( \sigma^2_{\nu_s} - 2 \sigma_{\nu_s} + \bar{q}_i \right) \left( \frac{1 - 1}{n} \right) + \sigma^2 \left( \frac{1 - 1}{N} \right) \\
+ \left( \frac{1}{n} \bar{q}_i - 2 \bar{q}_s + n \bar{q}_3 \right) + \left[ \left( \mu_{\nu_s} - n \mu_{\nu_s} \right) - \mu \right]^2
\]

or
\[
\text{var}_{\xi_i} \left( \hat{T} - T \right) = \left( \sigma^2_{\nu_s} - 2 \sigma_{\nu_s} \right) \left( \frac{1 - 1}{n} \right) + \left( \frac{1 - 1}{n} \right) \left( \frac{1 - 1}{N} \right) \sigma^2 + \\
\bar{q}_i \left( \frac{1 - 1}{n} \right) + \left( \frac{1}{n} \bar{q}_i - 2 \bar{q}_s + n \bar{q}_3 \right) + \left[ \left( \mu_{\nu_s} - n \mu_{\nu_s} \right) - \mu \right]^2
\]

Now \( q_{1s} = k^*_s \left( M_w s \right)^2 \left( \frac{1 - f_s}{f_s} \right) \sigma^2_s \) simplifies when predicting a PSU mean to
\[
q_{1s} = k^*_s \left( \frac{1 - f_s}{f_s} \right) \sigma^2_s M_s. \\
\text{Similarly, } q_{2s} \text{ and } q_{3s} \text{ simplify to } q_{2s} = \alpha s k^*_s \left( \frac{1 - f_s}{f_s} \right) \sigma^2_s M_s \text{ and } q_{3s} = \alpha s \left( \frac{1 - f_s}{f_s} \right) \sigma^2_s M_s. \]

Using these expressions,
\[
\bar{q}_i \left( \frac{1 - 1}{n} \right) + \left( \frac{1}{n} \bar{q}_i - 2 \bar{q}_s + n \bar{q}_3 \right) = \frac{1}{N} \sum_{s=1}^{N} \left[ k^*_s - 2 \alpha s k^*_s + n \alpha^2 s \right] \left( \frac{1 - f_s}{f_s} \right) \sigma^2_s M_s.
\]

When we are predicting a sample PSU mean, \( \alpha_s = \frac{1}{n} \left( k^* - 1 \right) \). As a result,
\[
k^*_s - 2 \alpha s k^*_s + n \alpha^2 s = k^*_s - 2 \left( \frac{1}{n} \bar{q}_s - 1 \right) k^*_s + \left( k^*_s - 1 \right) \frac{1}{n} \left( k^*_s - 1 \right)
\]
\[
= k^*_s + \frac{1}{n} \left( k^*_s - 1 \right) \left[ -2 k^*_s + k^*_s - 1 \right]
\]
\[
= k^*_s - \frac{1}{n} \left( k^*_s - 1 \right) \left[ k^*_s + 1 \right]
\]
\[
= k^*_s \left( 1 - \frac{1}{n} \right) + \frac{1}{n}
\]
so that
\[
\bar{q}_i \left( \frac{1 - 1}{n} \right) + \left( \frac{1}{n} \bar{q}_i - 2 \bar{q}_s + n \bar{q}_3 \right) = \frac{1}{N} \sum_{s=1}^{N} \left[ \left( 1 - \frac{1}{n} \right) k^*_s + \frac{1}{n} \left( 1 - f_s \right) \right] \sigma^2_s M_s.
\]

Finally, notice that
\[
\mu_{\nu_s} - n \mu_{\nu_s} = \frac{1}{N} \sum_{s=1}^{N} k^*_s \mu_s - \frac{n}{N} \sum_{s=1}^{N} \alpha_s \mu_s
\]
\[
= \frac{1}{N} \sum_{s=1}^{N} \left( k^*_s - n \alpha_s \right) \mu_s
\]
and that since \( \alpha_s = \frac{1}{n} \left( k^*_s - 1 \right) \), \( k^*_s - n \alpha_s = k^*_s - \left( k^*_s - 1 \right) = 1 \), so that \( \mu_{\nu_s} - n \mu_{\nu_s} = \frac{1}{N} \sum_{s=1}^{N} \mu_s = \mu \).
Using these results,

\[
\text{var}_{\hat{\gamma}_2} (\hat{T} - T) = (\sigma_{kr}^2 - 2\sigma_{rk,r} + \sigma^2) \left( 1 - \frac{1}{n} \right) + \frac{1}{N} \sum_{s=1}^{N} \left[ \left( 1 - \frac{1}{n} \right) k_s^{* 2} + \frac{1}{n} \left( 1 - f_s \right) \right] \sigma_s^2 M_s.
\]

This result checks the result in c07ed11.doc, and matches the simulation results given in c07ed09.doc.

**Summary Using \( \alpha \)**

In general, we express the EMSE as

\[
\text{var}_{\hat{\gamma}_2} (\hat{T} - T) = n \left( 1 - \frac{n}{N} \right) \left( \sigma_{kr}^2 \bar{c}_i^2 - 2\bar{c}_i \sigma_{rk,ra} + \sigma_{ar}^2 \right) + \left( \sigma_{kr}^2 - 2\sigma_{rk,r} + \bar{q}_i \right) \left( \sum_{s=1}^{n} (c_i - \bar{c}_i)^2 \right)
\]

\[+ n \left( \bar{c}_i^2 \bar{q}_i - 2\bar{q}_i \bar{c}_i + \bar{q}_i \right) + \left[ n (\bar{c}_i \mu_{kr} - \mu_{ar}) - N\bar{c}_i \mu_r \right] + 2n (\bar{c}_i - \bar{c}) (\sigma_{ra} - \bar{c}_i \sigma_{rk,r}) \bar{c}_i
\]

\[+ \sigma_s^2 \left( \sum_{i=1}^{N} (c_i - \bar{c})^2 \right) - \left( \frac{1}{N} \sum_{s=1}^{N} \left( n\bar{c}_i k_s^* - n\alpha_s - N\bar{c} \right) \right) r_s^2 \]

Terms in this expression are defined by

\[\sigma_{kr,ra} = \frac{1}{N-1} \sum_{s=1}^{N} (k_s r_s - \mu_{kr}) (\alpha_s r_s - \mu_{ar}), \quad \sigma_{kr,r} = \frac{1}{N-1} \sum_{s=1}^{N} (k_s r_s - \mu_{kr}) (r_s - \mu_r) \text{ and } \sigma_{r,ra} = \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r) (\alpha_s r_s - \mu_{ar}) \]

where

\[r_s = M_s w_s \mu_s, \quad \alpha_s = \bar{c}_i (k_s^* - 1) - \frac{n}{n} c_i \mu_s, \quad k_s^* = k_s - \frac{k_s}{M_s w_s \mu_s} \frac{1}{N} \sum_{s=1}^{N} \left[ \frac{1 - k_s}{1 \sum_{s=1}^{N} (1 - k_s)} \right] M_s w_s \mu_s \]

and

\[k_s = \frac{\mu_s^2}{\mu_s^2 + (1 - f_s) \left( \frac{N-1}{N} \right) \sigma_s^2 M_s}. \quad \text{Also, } \mu_r = \frac{1}{N} \sum_{s=1}^{N} r_s, \quad \mu_{kr} = \frac{1}{N} \sum_{s=1}^{N} k_s r_s, \quad \mu_{ar} = \frac{1}{N} \sum_{s=1}^{N} \alpha_s r_s, \quad \text{and} \]

\[q_{1s} = k_s^* \left( M_s w_s \right)^2 \left( \frac{1 - f_s}{f_s} \frac{\sigma_s^2}{M_s} \right), \quad q_{2s} = \alpha_s k_s^* \left( M_s w_s \right)^2 \left( \frac{1 - f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \text{ and} \]

\[q_{3s} = \alpha_s^2 \left( M_s w_s \right)^2 \left( \frac{1 - f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \text{ where } \bar{q}_1 = \frac{1}{N} \sum_{s=1}^{N} q_{1s}, \quad \bar{q}_2 = \frac{1}{N} \sum_{s=1}^{N} q_{2s}, \text{ and } \bar{q}_3 = \frac{1}{N} \sum_{s=1}^{N} q_{3s}. \text{ Finally,} \]

\[\bar{c}_i = \frac{1}{n} \sum_{i=1}^{n} c_i \text{ and } \bar{c} = \frac{1}{N} \sum_{i=1}^{N} c_i. \]

In the special case where we are predicting a sample PSU mean, the EMSE simplifies to

\[
\text{var}_{\hat{\gamma}_2} (\hat{T} - T) = (\sigma_{kr}^2 - 2\sigma_{rk,r} + \sigma^2) \left( 1 - \frac{1}{n} \right) + \frac{1}{N} \sum_{s=1}^{N} \left[ \left( 1 - \frac{1}{n} \right) k_s^{* 2} + \frac{1}{n} \left( 1 - f_s \right) \right] \sigma_s^2 M_s.
\]
Summary without Using $\alpha$

Notice that we can express $\alpha$ as

$$\alpha_s = \bar{c}_i \left(k_s^* - 1\right) - \frac{N - n}{n} \bar{c}_H$$

$$= \bar{c}_i k_s^* - \frac{n \bar{c}_i + (N - n) \bar{c}_H}{n}$$

$$= \bar{c}_i k_s^* - \frac{N \bar{c}}{n}$$

Using these results and the definitions of $\mu_r = \frac{1}{N} \sum_{s=1}^{N} r_s$ and $\mu_{kr} = \frac{1}{N} \sum_{s=1}^{N} k_s^* r_s$,

$$\mu_{ar} = \frac{1}{N} \sum_{s=1}^{N} \alpha_s r_s$$

$$= \frac{1}{N} \sum_{s=1}^{N} r_s \left(\bar{c}_i k_s^* - \frac{N \bar{c}}{n}\right)$$

$$= \bar{c}_i \left(\frac{1}{N} \sum_{s=1}^{N} k_s^* r_s\right) - \frac{n \bar{c}}{n} \left(\frac{1}{N} \sum_{s=1}^{N} r_s\right)$$

$$= \bar{c}_i \mu_{kr} - \frac{N \bar{c}}{n} \mu_r$$

Then $\alpha_s r_s - \mu_{ar} = \bar{c}_i \left(k_s^* r_s - \mu_{kr}\right) - \frac{N \bar{c}}{n} \left(r_s - \mu_r\right)$, and hence

$$\sigma_{ar}^2 = \frac{1}{N - 1} \sum_{s=1}^{N} \left(\alpha_s r_s - \mu_{ar}\right)^2$$

$$= \bar{c}_i^2 \frac{1}{N - 1} \sum_{s=1}^{N} \left(k_s^* r_s - \mu_{kr}\right)^2 - 2 \bar{c}_i \frac{N \bar{c}}{n} \frac{1}{N - 1} \sum_{s=1}^{N} \left(k_s^* r_s - \mu_{kr}\right) \left(r_s - \mu_r\right) + \left(\frac{N \bar{c}}{n^2}\right) \frac{1}{N - 1} \sum_{s=1}^{N} \left(r_s - \mu_r\right)^2$$

$$= \bar{c}_i^2 \sigma_{kr}^2 - 2 \bar{c}_i \frac{N \bar{c}}{n} \sigma_{kr,r} + \frac{\left(N \bar{c}\right)^2}{n^2} \sigma_r^2$$

while

$$\sigma_{rk,ar} = \frac{1}{N - 1} \sum_{s=1}^{N} \left(k_s^* r_s - \mu_{kr}\right) \left(\alpha_s r_s - \mu_{ar}\right)$$

$$= \bar{c}_i \frac{1}{N - 1} \sum_{s=1}^{N} \left(k_s^* r_s - \mu_{kr}\right)^2 - \frac{N \bar{c}}{n} \frac{1}{N - 1} \sum_{s=1}^{N} \left(k_s^* r_s - \mu_{kr}\right) \left(r_s - \mu_r\right)$$

$$= \bar{c}_i \sigma_{kr}^2 - \frac{N \bar{c}}{n} \sigma_{rk,r}$$

and
\[ \sigma_{r,ra} = \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)(\alpha_{r_s} - \mu_{ar}) \]
\[ = \bar{c}_j \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)(k_s^*r_s - \mu_{kr}) - \frac{N \bar{c}}{n} \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)^2. \]
\[ = \bar{c}_j \sigma_{rk,r} - \frac{N \bar{c}}{n} \sigma_r^2 \]

We also use \( \alpha_s = \bar{c}_i k_s^* - \frac{N \bar{c}}{n} \) to \( q_{2s} \) and \( q_{3s} \) in terms of \( q_{1s} = k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \),

\[ q_{2s} = \alpha_s k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \]
\[ = \left( \bar{c}_i k_s^* - \frac{N \bar{c}}{n} \right) k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \]
\[ = \bar{c}_i k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) - \frac{N \bar{c}}{n} k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \]
\[ = \bar{c}_i q_{1s} = \frac{N \bar{c}}{n} k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \]

and

\[ q_{3s} = \alpha_s^2 (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \]
\[ = \bar{c}_i^2 k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) - 2 \frac{N \bar{c}}{n} k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \]
\[ = \bar{c}_i^2 q_{1s} - 2 \frac{N \bar{c}}{n} k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \]

. Using these expressions, \( \bar{q}_2 = \bar{c}_i \bar{q}_i - \frac{N \bar{c}}{n} \sum_{s=1}^{N} k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \) and

\[ \bar{q}_3 = \bar{c}_i^2 \bar{q}_i - 2 \frac{N \bar{c}}{n} \left[ \frac{1}{N} \sum_{s=1}^{N} k_s^* (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right) \right] + \frac{(N \bar{c})^2}{n^2} \frac{1}{N} \sum_{s=1}^{N} (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma_s^2}{M_s} \right). \]

We use the expressions for \( \alpha_s = \bar{c}_i k_s^* - \frac{N \bar{c}}{n} , \mu_{ar} = \bar{c}_i \mu_{kr} - \frac{N \bar{c}}{n} \mu_r , \)

\[ \sigma_{rk,ra} = \bar{c}_i \sigma_{kr}^2 - \frac{N \bar{c}}{n} \sigma_{rk,r} , \sigma_{ar} = \bar{c}_i \sigma_{kr}^2 - 2 \frac{\bar{c}_i N \bar{c}}{n^2} \sigma_{kr,r} + \frac{(N \bar{c})^2}{n^2} \sigma_r^2 , \sigma_{r,ra} = \bar{c}_i \sigma_{rk,r} - \frac{N \bar{c}}{n} \sigma_r^2 , \]
\[ \bar{q}_2 = \bar{c}_i \bar{q}_i - \frac{N\bar{c}}{n} \frac{1}{N} \sum_{s=1}^{N} k_s^* (M_s w_s)^2 \left( \frac{1 - f_s}{f_s} \right) m_s^2 \]

and

\[ \bar{q}_3 = \bar{c}_i^2 \bar{q}_i - 2 \frac{N\bar{c}_i \bar{c}}{n} \frac{1}{N} \sum_{s=1}^{N} k_s^* (M_s w_s)^2 \left( \frac{1 - f_s}{f_s} \right) m_s^2 + \frac{(N\bar{c})^2}{n^2} \frac{1}{N} \sum_{s=1}^{N} (M_s w_s)^2 \left( \frac{1 - f_s}{f_s} \right) m_s^2 \]

to form equivalent expressions for terms in the EMSE. First,

\[
\left( \sigma^2_i c_i^2 - 2 \bar{c}_i \sigma_{rk,ar} + \sigma^2_{ar} \right) = \sigma^2_i c_i^2 - 2 \bar{c}_i \left( \sigma^2_i + \frac{N\bar{c}}{n} \sigma_{rk,r} \right) + \sigma^2_i \sigma^2_{kr} - 2 \bar{c}_i \sigma^2_{kr} - 2 \frac{\bar{c}_i N\bar{c}}{n} \sigma_{kr,r} + \frac{(N\bar{c})^2}{n^2} \sigma^2_r
\]

\[= \bar{c}_i^2 \sigma^2_{kr} - 2 \bar{c}_i^2 \sigma^2_{kr} + 2 \frac{\bar{c}_i N\bar{c}}{n} \sigma_{kr,r} + \bar{c}_i^2 \sigma^2_{kr} - 2 \frac{\bar{c}_i N\bar{c}}{n} \sigma_{kr,r} + \frac{(N\bar{c})^2}{n^2} \sigma^2_r .
\]

Next,

\[ \bar{c}_i \bar{q}_i - 2 \bar{q}_i \bar{c}_i + \bar{q}_3 = \bar{c}_i^2 \bar{q}_i - 2 \bar{c}_i^2 \bar{q}_i + 2 \frac{\bar{c}_i N\bar{c}}{n} \left[ \frac{1}{N} \sum_{s=1}^{N} k_s^* (M_s w_s)^2 \left( \frac{1 - f_s}{f_s} \right) m_s^2 \right] + \frac{(N\bar{c})^2}{n^2} \frac{1}{N} \sum_{s=1}^{N} (M_s w_s)^2 \left( \frac{1 - f_s}{f_s} \right) m_s^2 \]

\[= \frac{(N\bar{c})^2}{n^2} \frac{1}{N} \sum_{s=1}^{N} (M_s w_s)^2 \left( \frac{1 - f_s}{f_s} \right) m_s^2 \]

, and

\[ n(\bar{c}_i \mu_{kr} - \mu_{ar}) - N\bar{c} \mu_r = n \left( \bar{c}_i \mu_{kr} - \bar{c}_i \mu_{kr} + \frac{N\bar{c}}{n} \mu_r \right) - N\bar{c} \mu_r .
\]

\[= 0 \]

In addition,

\[ \sigma_{r,ra} - \bar{c}_i \sigma_{rk,r} = \bar{c}_i \sigma_{rk,r} - \frac{N\bar{c}}{n} \sigma^2_r - \bar{c}_i \sigma_{rk,r} \]

\[= - \frac{N\bar{c}}{n} \sigma^2_r \]

and

\[ \left( n\bar{c}_i k_s^* - n\alpha_s - N\bar{c} \right) = n\bar{c}_i k_s^* - n \left( \bar{c}_i k_s^* - \frac{N\bar{c}}{n} \right) - N\bar{c} \]

\[= n\bar{c}_i k_s^* - n\bar{c}_i k_s^* + N\bar{c} - N\bar{c} \]

\[= 0 \]
In summary, 
\[
\left( \sigma_{kr}^2 \bar{c}_i^2 - 2 \bar{c}_i \sigma_{rk,ra} + \sigma_{ar}^2 \right) = \frac{(N \bar{c})^2}{n^2} \sigma_r^2,
\]
\[
\bar{c}_i \bar{q}_i - 2 \bar{q}_i \bar{c}_i + \bar{q}_3 = \frac{(N \bar{c})^2}{n^2} \frac{1}{N} \sum_{s=1}^{N} (M_s w_s) \left( \frac{1 - f_s}{f_s} \sigma_s^2 \right), \quad n(\bar{c}_i \mu_{kr} - \mu_{ar}) - N \bar{c} \mu_r = 0,
\]
\[
\sigma_{r,ra} - \bar{c}_i \sigma_{rk,r} = -\frac{N \bar{c}}{n} \sigma_r^2,
\]
and \( (n \bar{c}_i k^*_s - n \alpha_s - N \bar{c}) = 0 \). As a result, substituting these expressions into the EMSE given by 
\[
\text{var}_{\hat{\xi}_N} (\hat{T} - T) = n \left( 1 - \frac{n}{N} \left( \sigma_{kr}^2 - 2 \bar{c}_i \sigma_{rk,ra} + \sigma_{ar}^2 \right) + \left( \sigma_{kr}^2 - 2 \sigma_{rk,r}^2 + \bar{q}_1 \right) \left( \sum_{i=1}^{n} (c_i - \bar{c}_i)^2 \right) - \frac{1}{N} \sum_{i=1}^{n} \left( n \bar{c}_i k^*_s - n \alpha_s - N \bar{c} \right) r_s \right)^2.
\]
we find that 
\[
\text{var}_{\hat{\xi}_N} (\hat{T} - T) = n \left( 1 - \frac{n}{N} \left( \sigma_{kr}^2 - 2 \bar{c}_i \sigma_{rk,ra} + \sigma_{ar}^2 \right) + \left( \sigma_{kr}^2 - 2 \sigma_{rk,r}^2 + \bar{q}_1 \right) \left( \sum_{i=1}^{n} (c_i - \bar{c}_i)^2 \right) - \frac{1}{N} \sum_{i=1}^{n} \left( n \bar{c}_i k^*_s - n \alpha_s - N \bar{c} \right) r_s \right)^2.
\]
Grouping terms,
\[
\text{var}_{\hat{\xi}_N} (\hat{T} - T) = \left( \sigma_{kr}^2 - 2 \sigma_{rk,r} + \bar{q}_1 \right) \left( \sum_{i=1}^{n} (c_i - \bar{c}_i)^2 \right) + \left( \sigma_{kr}^2 - 2 \sigma_{rk,r} + \bar{q}_1 \right) \left( \sum_{i=1}^{n} (c_i - \bar{c}_i)^2 \right) - \frac{1}{N} \sum_{i=1}^{n} \left( n \bar{c}_i k^*_s - n \alpha_s - N \bar{c} \right) r_s.
\]
where 
\[
\sigma_{kr}^2 = \frac{1}{N-1} \sum_{s=1}^{N} (k^*_s r_s - \mu_{kr})^2, \quad \sigma_r^2 = \frac{1}{N-1} \sum_{s=1}^{N} (r_s - \mu_r)^2, \quad \sigma_{rk,r} = \frac{1}{N-1} \sum_{s=1}^{N} (k^*_s r_s - \mu_{kr})(r_s - \mu_r),
\]
and \( \bar{q}_1 = \frac{1}{N} \sum_{s=1}^{N} k^*_s (M_s w_s)^2 \left( \frac{1 - f_s}{f_s} \sigma_s^2 \right) \) where \( r_s = M_s w_s \mu_s \),
\[
k^*_s = k_s - \frac{k_s}{M_s w_s \mu_s} \frac{1}{N} \sum_{s=1}^{N} \left[ \frac{1}{N} \sum_{s=1}^{N} (1 - k_s) \right] M_s w_s \mu_s, \quad \text{and} \quad k_s = \frac{\mu_r^2}{\mu_r^2 + (1 - f_s) \frac{(N-1)}{N} \frac{\sigma_r^2}{\sigma_s^2}}.
\]
Also, 
\[
\mu_r = \frac{1}{N} \sum_{s=1}^{N} r_s, \quad \mu_{kr} = \frac{1}{N} \sum_{s=1}^{N} k^*_s r_s, \quad \bar{c}_i = \frac{1}{n} \sum_{i=1}^{n} c_i, \quad \text{and} \quad \bar{c} = \frac{1}{N} \sum_{i=1}^{n} c_i.
\]
In other documents, we have used \( d_s = M_s w_s \mu_s \) instead of \( r_s = M_s w_s \mu_s \). Also, we have defined \( v_{se}^2 = (M_s w_s)^2 \left( \frac{1-f_s}{f_s} \frac{\sigma^2}{M_s} \right) \). As a result, \( \bar{q}_1 = \frac{1}{N} \sum_{s=1}^{N} k_s^2 v_{se}^2 \). Replacing \( r_s \) by \( d_s \),

\[
\text{var}_{N} \left( \hat{T} - T \right) = \left( \sum_{i=1}^{n} (c_i - \bar{c}_i)^2 \right) \left[ \sigma^2_{kd} - 2\sigma_{kd,d} + \frac{1}{N} \sum_{s=1}^{N} k_s^2 v_{se}^2 \right] + \frac{\left( \frac{N}{n} \right)^2}{n} \left( \frac{1}{N} \sum_{s=1}^{N} v_{se}^2 \right) + \frac{\sum_{i=1}^{n} (c_i - \bar{c}_i)^2}{n} m_x \sigma^2 \]

where \( \sigma^2_{kd} = \frac{1}{N-1} \sum_{s=1}^{N} (k_s d_s - \mu_{kd})^2 \), \( \sigma^2_d = \frac{1}{N-1} \sum_{s=1}^{N} (d_s - \mu_d)^2 \), \( \sigma_{kd,d} = \frac{1}{N-1} \sum_{s=1}^{N} (k_s d_s - \mu_{kd})(d_s - \mu_d) \), where \( d_s = M_s w_s \mu_s \),

\[
k_s = k_s - \frac{k_s}{d_s} \frac{1}{N} \sum_{x=1}^{N} \left( \frac{1}{\sum_{x=1}^{N} (1-k_s)} \right) d_x \quad \text{and} \quad k_i = \frac{\mu_i^2}{\mu_i^2 + (1-f_i) \left( \frac{N-1}{N} \right) \frac{\sigma^2}{m_x}}
\]

Also, \( \mu_d = \frac{1}{N} \sum_{s=1}^{N} d_s \),

\[
\mu_{kd} = \frac{1}{N} \sum_{s=1}^{N} k_s^2 d_s \quad \text{and} \quad \bar{c}_i = \frac{1}{n} \sum_{i=1}^{n} c_i \quad \text{and} \quad \bar{c} = \frac{1}{N} \sum_{i=1}^{n} c_i .
\]

**Simplifications when Predict a Sample PSU Mean**

If the PSU is in the sample, \( \bar{c}_i = \frac{1}{n} \), \( \sum_{i=1}^{n} (c_i - \bar{c}_i)^2 = \left( \frac{n-1}{n} \right) \), \( \bar{c} = \frac{1}{N} \) and \( \sum_{i=1}^{n} (c_i - \bar{c}_i)^2 = \frac{N-1}{N} \). Also, \( w_s = \frac{1}{M_s} \) so that \( d_s = \mu_s \) and \( \sigma^2_d = \sigma^2 \). As a result,

\[
\text{var}_{N} \left( \hat{T} - T \right) = \left( \sigma^2_{kd} - 2\sigma_{kd,d} + \frac{1}{N} \sum_{s=1}^{N} k_s^2 v_{se}^2 \right) \left( \frac{n-1}{n} \right) + \frac{1}{n} \left( \frac{1}{N} \sum_{s=1}^{N} v_{se}^2 \right) + \frac{\left( \frac{N-1}{n} \right)^2}{n} \left( \frac{1}{N} \sum_{s=1}^{N} v_{se}^2 \right) + \frac{\sum_{i=1}^{n} (c_i - \bar{c}_i)^2}{n} m_x \sigma^2
\]

or

\[
\text{var}_{N} \left( \hat{T} - T \right) = \left( \frac{n-1}{n} \right) \left( \sigma^2_{kd} - 2\sigma_{kd,d} + \frac{1}{N} \sum_{s=1}^{N} k_s^2 v_{se}^2 \right) + \frac{1}{n} \left( \frac{1}{N} \sum_{s=1}^{N} v_{se}^2 \right) + \frac{\left( \frac{N-1}{n} \right)^2}{n} \left( \frac{1}{N} \sum_{s=1}^{N} v_{se}^2 \right) .
\]
If the PSU is not in the sample, \( \bar{c}_i = 0 \), \( \sum_{i=1}^{n_i} (c_i - \bar{c})^2 = 0 \), \( \bar{c} = \frac{1}{N} \) and
\[
\sum_{i=1}^{N} (c_i - \bar{c})^2 = \frac{N-1}{N}. \]
As a result,
\[
\text{var}_{\text{4,55}} (\hat{T} - T) = n \left[ \frac{1}{N} \sum_{i=1}^{N} v_{se}^2 \right] + \left[ \left( 1 - \frac{n}{N} \right) + \frac{2}{N} + \left( 1 - \frac{1}{N} \right) \right] \sigma^2
\]
\[
= n \left[ \frac{1}{N} \sum_{i=1}^{N} v_{se}^2 \right] + \left[ \frac{1}{n} \left( 1 - \frac{n}{N} \right) + \left( 1 + \frac{1}{N} \right) \right] \sigma^2
\]
or
\[
\text{var}_{\text{4,55}} (\hat{T} - T) = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1 - f_s}{f_s} \right) \frac{\sigma^2}{M_i} + \left[ \frac{1}{n} \left( 1 - \frac{n}{N} \right) + \left( 1 + \frac{1}{N} \right) \right] \sigma^2.
\]