Predictors for Simple 1-Factor Completely Randomized Design with Permutation Treatments

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Introduction

We sketch the development of predictors of treatment parameters in a simple 1 way completely randomized design using a random permutation model prediction framework. The context we consider is for a finite population with labeled units indexed by $s = 1,...,N$. We assume that treatments are indexed by $a = 1,...,A$. The experiment consists of randomly assigning a simple random sample of n subjects to each of the A treatments (where $nA \le N$) We permute the treatments prior to assigning a treatment to a subject. The goal is to predict the mean of a realized treatment.

The Population

We use the potentially observable population concept of Rubin (2005) in formulating the problem. We assume that each unit could be potentially observed under each of the treatments, and represent the response for unit s given treatment a by the non-stochastic value y_{sa}. The mean for treatment a is defined by $\mu_a = \frac{1}{N}\sum_{s=1}^{N} y_{sa}$. The mean for subject s is defined by $\mu_s = \frac{1}{A}\sum_{a=1}^{A} y_{sa}$. The overall mean over all the potential observable values is $\mu = \frac{1}{NA}\sum_{a=1}^{A}\sum_{s=1}^{N} y_{sa}$.

Parameterization

We define additional parameters in terms of these basic values. First, define $\beta_s = \mu_s - \mu$ as the effect of unit s, $\alpha_a = \mu_a - \mu$ as the effect of treatment a, ε_{sa} as the interaction between unit s and effect of treatment a. As a result,

$$y_{sa} = \mu + \beta_s + \alpha_a + \varepsilon_{sa} \qquad (1).$$

We represent the $N \times 1$ potentially observable responses for treatment a by $\mathbf{y}_a$ where $\mathbf{y}_a = \begin{pmatrix} y_{1a} & y_{2a} & \dots & y_{Na} \end{pmatrix}'$, and the $N \times A$ matrix of potentially observable responses as $\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_A \end{pmatrix}$. The vector of treatment parameters is given by $\boldsymbol{\mu} = \left(\frac{1}{N}\mathbf{1}_N'\mathbf{y}\right)'$ where $\mathbf{1}_N$ is $N \times 1$ vector with all elements equal to one. We can express $\mathbf{y}$ in terms of μ, $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_N \end{pmatrix}'$, $\boldsymbol{\tau} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_A \end{pmatrix}'$ and $\boldsymbol{\varepsilon} = \left(\left(\varepsilon_{sa}\right)\right)$:

$$\mathbf{y} = \mathbf{1}_N\mathbf{1}_A'\mu + \boldsymbol{\beta}\,\mathbf{1}_A' + \mathbf{1}_N\boldsymbol{\tau}' + \boldsymbol{\varepsilon} \qquad (2)$$

We define the co-variance matrix of the response for treatments as $\frac{N-1}{N}\mathbf{\Sigma}$, where $\mathbf{\Sigma}=\left(\left(\sigma_{aa*}\right)\right)_{A\times A}=\frac{1}{N-1}\mathbf{y}'\mathbf{P}_N\mathbf{y}$ and $\sigma_{aa*}=\frac{1}{N-1}\sum_{s=1}^{N}\left(y_{sa}-\mu_a\right)\left(y_{sa*}-\mu_{a*}\right)$ for all $a=1,...,A$ and $a*=1,...,A$. We define $\mathbf{P}_K=\mathbf{I}_K-\frac{1}{K}\mathbf{J}_K$ where $\mathbf{I}_K$ is $K\times K$ identity matrix and $\mathbf{J}_K=\mathbf{1}_K\mathbf{1}_K'$.

Further, we define $\bar{\sigma}^2=\frac{1}{A}tr\left(\mathbf{\Sigma}\right)=\frac{\sum_{a=1}^{A}\sigma_{aa}}{A}$. Similarly, we define the variance of the response for unit s as $\frac{A-1}{A}\sigma_{ss}$ where $\sigma_{ss}=\frac{1}{A-1}\sum_{a=1}^{A}\left(y_{sa}-\mu_s\right)^2$, and the covariance for unit s and unit $s*$ as $\frac{A-1}{A}\sigma_{ss*}$, where $\sigma_{ss*}=\frac{1}{A-1}\sum_{a=1}^{A}\left(y_{sa}-\mu_s\right)\left(y_{s*a}-\mu_{s*}\right)$, and summarizing the covariances as $\frac{A-1}{A}\mathbf{\Sigma}_A$ where $\mathbf{\Sigma}_A=\left(\left(\sigma_{ss*}\right)\right)=\frac{1}{A-1}\mathbf{y}\mathbf{P}_A\mathbf{y}'$.

Sampling

An experiment will result in observing the response for subjects assigned to each treatment. Suppose that a simple random sample of nA subjects is selected without replacement and placed into A consecutive treatment groups of n subjects, indexing the treatment groups by $j=1,...,A$ and the selected subjects by $i=1,...,N$ in each treatment group. We will refer to the subject in position i as PSU i. Permute the treatments and index the treatments in the permutation by $j=1,...,A$ corresponding to treatment groups. One way to represent the sampling is to think of all possible permutations of the units and all possible permutations of treatments. We represent the experiment by a set of random variables that represent the potentially observed units.

Permutation of the population

Let $i=1,...,N$ index the position of a subject in a permutation of subjects, $s=1,...N$ be the label of a subject. The permutation of units can be defined in terms of U_{is}. Explicitly, the random variable U_{is} takes on a value of one if the subject s is assigned to position i in a permutation, or zero otherwise. The matrix of indicator random variables for units is given by $\mathbf{U}_{N\times N}=\left(\left(U_{is}\right)\right)$. Similarly, let $j=1,...,A$ index the position of a treatment in a permutation of treatments, $a=1,...,A$ labels the treatments. The permutation of the treatments can be defined in terms of indicator random variable V_{ja}, where V_{ja} takes on a value of one if the treatment a is assigned to position j in a permutation, or zero otherwise. The matrix of indicator random variables for treatments is $\mathbf{V}_{A\times A}=\left(\left(V_{ja}\right)\right)$.

We use $\mathbf{U}$ and $\mathbf{V}$ to represent a joint permutation of units and treatments. Explicitly, let $\mathbf{U} = \begin{pmatrix}\mathbf{U}_1 & \mathbf{U}_2 & \cdots & \mathbf{U}_N\end{pmatrix}' = \left(\left(\mathbf{U}_i\right)\right)'$ and $\mathbf{V} = \begin{pmatrix}\mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_A\end{pmatrix}' = \left(\left(\mathbf{V}_j\right)\right)'$. Then $Y_{ij} = \mathbf{U}_i'\mathbf{y}\mathbf{V}_j = \sum_{s=1}^{N}\sum_{a=1}^{A} U_{is}V_{ja}y_{sa}$. Using the property of the permutation matrix, for any permutation matrix, we get

$$E\left(\mathbf{U}\right) = \frac{1}{N}\mathbf{J}_N$$

$$Var_U\left[vec\left(\mathbf{U}\right)\right] = \frac{1}{N-1}\left(\mathbf{P}_N \otimes \mathbf{P}_N\right)$$

$$Var\left(\mathbf{U}_i\right) = \frac{1}{N}\mathbf{P}_N$$

$$Cov\left(\mathbf{U}_i\mathbf{U}_{i*}\right) = -\frac{1}{N\left(N-1\right)}\mathbf{P}_N \qquad \text{when } i \neq i*$$

$$E\left(\mathbf{U}_i\mathbf{U}_{i*}\right) = \frac{1}{N}\mathbf{I}_N \text{ when } i = i* \ (3)$$

$$E\left(\mathbf{U}_i\mathbf{U}_{i*}\right) = \frac{1}{N\left(N-1\right)}\left(\mathbf{J}_N - \mathbf{I}_N\right) \text{ when } i \neq i*.(4)$$

We define the joint random permutation model for the population as $\mathbf{Y} = \left(\left(Y_{ij}\right)\right) = \mathbf{U}\mathbf{y}\mathbf{V}'$ where Y_{ij} indicates the response of PSU i in the treatment group.

Using (1),

$$\mathbf{Y} = \mathbf{U}\mathbf{1}_N\mathbf{1}_A'\mathbf{V}'\mu + \mathbf{U}\boldsymbol{\beta}\mathbf{1}_A'\mathbf{V}' + \mathbf{U}\mathbf{1}_N\boldsymbol{\tau}'\mathbf{V}' + \mathrm{E}, \qquad (5)$$

where $\mathbf{E} = \mathbf{U}\boldsymbol{\varepsilon}\mathbf{V}'$. Note that $\mathbf{U}\mathbf{1}_N = \mathbf{1}_N, \mathbf{1}_A'\mathbf{V}' = \mathbf{1}_A'$. As a result,

$$\mathbf{Y} = \mathbf{1}_N\mathbf{1}_A'\mu + \mathbf{U}\boldsymbol{\beta}\mathbf{1}_A' + \mathbf{1}_N\boldsymbol{\tau}'\mathbf{V}' + \mathbf{U}\boldsymbol{\varepsilon}\mathbf{V}'. \qquad (6)$$

We also define $\mathbf{B} = \mathbf{U}\boldsymbol{\beta}$ and $\mathbf{T} = \mathbf{V}\boldsymbol{\tau}$. Then

$$\mathbf{Y} = \mathbf{1}_N\mathbf{1}_A'\mu + \mathbf{B}\mathbf{1}_A' + \mathbf{1}_N\mathbf{T}' + \mathrm{E} \qquad (7)$$

We refer to this as the random one factor permutation model.

Expectation and Variance:

We determine the expected value and the variance of the random one factor permutation model next. We use $E_{UV}\left(\mathbf{Y}\right) = E_V\left(E_{U|V}\left(\mathbf{Y}\right)\right)$ to determine the expected value. Note that $E_{U|V}\left(\mathbf{Y}\right) = E_{U|V}\left(\mathbf{U}\mathbf{y}\mathbf{V}'\right) = \left[E_{U|V}\left(\mathbf{U}\right)\right]\mathbf{y}\mathbf{V}'$ where $E_U\left(\mathbf{U}\right) = \frac{1}{N}\mathbf{J}_N$ since $E_U\left(U_{is}\right) = \frac{1}{N}$ for all $i = 1,\ldots,N$ and $s = 1,\ldots N$. Further, $E_{UV}\left(\mathbf{Y}\right) = \frac{1}{N}\mathbf{J}_N\mathbf{y}E_V\left(\mathbf{V}'\right)$. Since $E_V\left(\mathbf{V}'\right) = \frac{1}{A}\mathbf{J}_A$ we find that $E_{UV}\left(\mathbf{Y}\right) = \frac{1}{AN}\mathbf{J}_N\mathbf{y}\mathbf{J}_A$. Using the definition of μ, $E_{UV}\left(\mathbf{Y}\right) = \mu\mathbf{1}_N\mathbf{1}_A'$.

We use a column expansion of $\mathbf{Y}$ to develop an expression for the variance.

$$Var_{UV}\left[vec(\mathbf{Y})\right]=Var_V\left(E_{U|V}\left[vec(\mathbf{Y})\right]\right)+E_V\left(Var_{U|V}\left[vec(\mathbf{Y})\right]\right)$$

We first simplify an expression for $Var_V\left(E_{U|V}\left[vec(\mathbf{Y})\right]\right)$.

Recall that $E_{U|V}\left[vec(\mathbf{Y})\right]=vec\left[E_{U|V}(\mathbf{Y})\right]$ and using equation (7), $E_{U|V}(\mathbf{Y})=E_{U|V}\left(\mathbf{1}_N\mathbf{1}'_A\mu+\mathbf{B}\mathbf{1}'_A+\mathbf{1}_N\mathbf{T}'+\mathrm{E}\right)$. Note that $\mathbf{1}_N\mathbf{1}'_A\mu$ and $\mathbf{1}_N\mathbf{T}'=\mathbf{1}_N\boldsymbol{\tau}'\mathbf{V}'$ are constants given $\mathbf{V}$. To evaluate $E_{U|V}\left(\mathbf{B}\mathbf{1}'_A\right)$, we substitute $\mathbf{B}=\mathbf{U}\boldsymbol{\beta}$. Then

$$\begin{aligned}E_{U|V}\left(\mathbf{B}\mathbf{1}'_A\right)&=E_{U|V}\left(\mathbf{U}\boldsymbol{\beta}\mathbf{1}'_A\right)\\&=E_{U|V}(\mathbf{U})\boldsymbol{\beta}\mathbf{1}'_A\\&=\frac{1}{N}\mathbf{J}_N\boldsymbol{\beta}\mathbf{1}'_A\\&=\frac{1}{N}\mathbf{1}_N\mathbf{1}'_N\boldsymbol{\beta}\mathbf{1}'_A\end{aligned}$$

Since $\mathbf{1}'_N\boldsymbol{\beta}=\sum_{s=1}^{N}\beta_s=0$, then $E_{U|V}\left(\mathbf{B}\mathbf{1}'_A\right)=0$

Since $\mathrm{E}=\mathbf{U}\boldsymbol{\varepsilon}\mathbf{V}'$, $E_{U|V}(\mathrm{E})=\left[E_{U|V}(\mathbf{U})\right](\boldsymbol{\varepsilon}\mathbf{V}')$ which simplifies to $E_{U|V}(\mathrm{E})=\frac{1}{N}\mathbf{1}_N\mathbf{1}'_N\boldsymbol{\varepsilon}\mathbf{V}'$. Let $\boldsymbol{\varepsilon}=\left((\boldsymbol{\varepsilon}_a)\right)$ where $\boldsymbol{\varepsilon}_a=\begin{pmatrix}\varepsilon_{1a} & \varepsilon_{2a} & \cdots & \varepsilon_{Na}\end{pmatrix}'$. Hence,

$\mathbf{1}'_N\boldsymbol{\varepsilon}_a=\sum_{s=1}^{N}\varepsilon_{sa}$, $\beta_s=\mu_s-\mu$, $\mu_a=\mu+\alpha_a$ and $\mu_a=\frac{1}{N}\sum_{s=1}^{N}y_{sa}$, hence

$$\begin{aligned}\mathbf{1}'_N\boldsymbol{\varepsilon}_a&=\sum_{s=1}^{N}\varepsilon_{sa}=\sum_{s=1}^{N}\left(y_{sa}-\mu-\beta_s-\alpha_a\right)\\&=\sum_{s=1}^{N}y_{sa}-\sum_{s=1}^{N}\beta_s-\sum_{s=1}^{N}(\mu+\alpha_a)\\&=N\mu_a-0-N\mu_a\\&=0\end{aligned}$$

As a result, $\mathbf{1}'_N\boldsymbol{\varepsilon}=\mathbf{0}$ and $E_{U|V}(\mathrm{E})=\mathbf{0}$. Hence, $E_{U|V}(\mathbf{Y})=\mathbf{1}_N\mathbf{1}'_A\mu+\mathbf{1}_N\mathbf{T}'$. Next, we evaluate

$$\begin{aligned}Var_V\left(E_{U|V}\left[vec(\mathbf{Y})\right]\right)&=Var_V\left[vec\left(\mathbf{1}_N\mathbf{1}'_A\mu+\mathbf{1}_N\mathbf{T}'\right)\right]\\&=Var_V\left[vec\left(\mathbf{1}_N\mathbf{T}'\right)\right]\end{aligned}$$

Substituting $\mathbf{T}=\mathbf{V}\boldsymbol{\tau}$,

$$\begin{aligned}vec\left(\mathbf{1}_N\mathbf{T}'\right)&=vec\left(\mathbf{1}_N\boldsymbol{\tau}'\mathbf{V}'\right)\\&=\left[\mathbf{I}_A\otimes\left(\mathbf{1}_N\boldsymbol{\tau}'\right)\right]vec(\mathbf{V}')\end{aligned}$$

As a result, $var_V\left(E_{U|V}\left[vec(\mathbf{Y})\right]\right)=\left[\mathbf{I}_A\otimes\left(\mathbf{1}_N\boldsymbol{\tau}'\right)\right]var_V\left[vec(\mathbf{V}')\right]\left[\mathbf{I}_A\otimes\left(\mathbf{1}_N\boldsymbol{\tau}'\right)\right]'$.

We evaluate $var_V\left[vec(\mathbf{V}')\right]$ similar to expression for the $Var_U\left[vec(\mathbf{U})\right]=\frac{1}{N-1}(\mathbf{P}_N\otimes\mathbf{P}_N)$, such that $Var_U\left[vec(\mathbf{V}')\right]=\frac{1}{A-1}(\mathbf{P}_A\otimes\mathbf{P}_A)$. As a result,

$$\begin{aligned}Var_V\left(E_{U|V}\left[vec(\mathbf{Y})\right]\right)&=\frac{1}{A-1}\left[\mathbf{I}_A\otimes(\mathbf{1}_N\boldsymbol{\tau}')\right](\mathbf{P}_A\otimes\mathbf{P}_A)\left[\mathbf{I}_A\otimes(\mathbf{1}_N\boldsymbol{\tau}')\right]'\\&=\mathbf{P}_A\otimes\mathbf{1}_N\left(\frac{1}{A-1}\boldsymbol{\tau}'\mathbf{P}_A\boldsymbol{\tau}\right)\mathbf{1}_N'\\&=\left(\frac{1}{A-1}\boldsymbol{\tau}'\mathbf{P}_A\boldsymbol{\tau}\right)(\mathbf{P}_A\otimes\mathbf{1}_N\mathbf{1}_N')\end{aligned}$$

We define $\sigma_A^2=\frac{1}{A-1}\boldsymbol{\tau}'\mathbf{P}_A\boldsymbol{\tau}$, and obtain

$$Var_V\left(E_{U|V}\left[vec(\mathbf{Y})\right]\right)=\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N \qquad (8)$$

Next, we evaluate $E_V\left(Var_{U|V}\left[vec(\mathbf{Y})\right]\right)$. First, using (6),

$$\begin{aligned}Var_{U|V}\left[vec(\mathbf{Y})\right]&=Var_{U|V}\left[vec\left(\mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{U}\boldsymbol{\beta}\mathbf{1}_A'+\mathbf{1}_N\boldsymbol{\tau}'\mathbf{V}'+\mathbf{U}\boldsymbol{\varepsilon}\mathbf{V}'\right)\right]\\&=Var_{U|V}\left[vec\left(\mathbf{U}\left[\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}'\right]\right)\right]\end{aligned}$$

since other terms do not depend on $\mathbf{U}$. Using the vec expansion,

$$vec\left(\mathbf{U}\left[\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}'\right]\right)=\left(\left[\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}'\right]'\otimes\mathbf{I}_N\right)vec(\mathbf{U}),$$

and using $Var_U\left[vec(\mathbf{U})\right]=\frac{1}{N-1}(\mathbf{P}_N\otimes\mathbf{P}_N)$,

$$\begin{aligned}Var_{U|V}\left[vec(\mathbf{Y})\right]&=\left[(\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}')'\otimes\mathbf{I}_N\right]Var_U\left[vec(\mathbf{U})\right]\left[(\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}')'\otimes\mathbf{I}_N\right]'\\&=\frac{1}{N-1}\left[(\mathbf{1}_A\boldsymbol{\beta}'+\mathbf{V}\boldsymbol{\varepsilon}')\otimes\mathbf{I}_N\right](\mathbf{P}_N\otimes\mathbf{P}_N)\left[(\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}')\otimes\mathbf{I}_N\right].\\&=\frac{1}{N-1}\left[(\mathbf{1}_A\boldsymbol{\beta}'+\mathbf{V}\boldsymbol{\varepsilon}')\mathbf{P}_N(\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}')\right]\otimes\mathbf{P}_N\end{aligned}$$

We can express $\mathbf{1}_A=\mathbf{V}\mathbf{1}_A$. As a result,

$$\begin{aligned}Var_{U|V}\left[vec(\mathbf{Y})\right]&=\frac{1}{N-1}\left[(\mathbf{1}_A\boldsymbol{\beta}'+\mathbf{V}\boldsymbol{\varepsilon}')\mathbf{P}_N(\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\mathbf{V}')\right]\otimes\mathbf{P}_N\\&=\frac{1}{N-1}\left[(\mathbf{V}\mathbf{1}_A\boldsymbol{\beta}'+\mathbf{V}\boldsymbol{\varepsilon}')\mathbf{P}_N(\boldsymbol{\beta}\mathbf{1}_A'\mathbf{V}'+\boldsymbol{\varepsilon}\mathbf{V}')\right]\otimes\mathbf{P}_N.\\&=\frac{1}{N-1}\left[\mathbf{V}(\mathbf{1}_A\boldsymbol{\beta}'+\boldsymbol{\varepsilon}')\mathbf{P}_N(\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon})\mathbf{V}'\right]\otimes\mathbf{P}_N\end{aligned}$$

Notice that

$$\begin{aligned}\mathbf{P}_N\mathbf{y} &= \mathbf{P}_N\mathbf{1}_N\mathbf{1}'_A\mu + \mathbf{P}_N\boldsymbol{\beta}\mathbf{1}'_A + \mathbf{P}_N\mathbf{1}_N\boldsymbol{\tau}' + \mathbf{P}_N\boldsymbol{\varepsilon} \\ &= \mathbf{P}_N\boldsymbol{\beta}\mathbf{1}'_A + \mathbf{P}_N\boldsymbol{\varepsilon} \\ &= \boldsymbol{\beta}\mathbf{1}'_A + \boldsymbol{\varepsilon}\end{aligned}.$$

Also, since $\mathbf{P}_N$ is idempotent, then $\mathbf{y}'\mathbf{P}_N\mathbf{y} = (\mathbf{P}_N\mathbf{y})'\mathbf{P}_N(\mathbf{P}_N\mathbf{y})$. Then, since $\boldsymbol{\Sigma} = \frac{1}{N-1}\mathbf{y}'\mathbf{P}_N\mathbf{y}$,

$\boldsymbol{\Sigma} = \frac{1}{N-1}(\mathbf{1}_A\boldsymbol{\beta}' + \boldsymbol{\varepsilon}')\mathbf{P}_N(\boldsymbol{\beta}\mathbf{1}'_A + \boldsymbol{\varepsilon})$. As a result,

$$Var_{U|V}\left[vec(\mathbf{Y})\right] = (\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}') \otimes \mathbf{P}_N .$$

The last step is to take the expectation of $Var_{U|V}\left[vec(\mathbf{Y})\right] = (\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}') \otimes \mathbf{P}_N$ with respect to $\mathbf{V}$.

$$E_V\left(Var_{U|V}\left[vec(\mathbf{Y})\right]\right) = \left[E_V(\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}')\right] \otimes \mathbf{P}_N$$

In order to get $E_V(\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}')$, recall that $\mathbf{V} = (\mathbf{V}_1 \quad \mathbf{V}_2 \quad \cdots \quad \mathbf{V}_A)' = ((\mathbf{V}_j))'$ where

$\mathbf{V}_j = (V_{j1} \quad V_{j2} \quad \ldots \quad V_{jA})'$, then

$$\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}' = \left(\begin{array}{c|c|c|c} \mathbf{V}_1'\boldsymbol{\Sigma}\mathbf{V}_1 & \mathbf{V}_1'\boldsymbol{\Sigma}\mathbf{V}_2 & \cdots & \mathbf{V}_1'\boldsymbol{\Sigma}\mathbf{V}_A \\ \hline \mathbf{V}_2'\boldsymbol{\Sigma}\mathbf{V}_1 & \mathbf{V}_2'\boldsymbol{\Sigma}\mathbf{V}_2 & \cdots & \mathbf{V}_2'\boldsymbol{\Sigma}\mathbf{V}_A \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{V}_A'\boldsymbol{\Sigma}\mathbf{V}_1 & \mathbf{V}_A'\boldsymbol{\Sigma}\mathbf{V}_2 & \cdots & \mathbf{V}_A'\boldsymbol{\Sigma}\mathbf{V}_A \end{array}\right)$$

To determine the expectation, we evaluate the expectation of each item of the matrix $E\left(\mathbf{V}_j'\boldsymbol{\Sigma}\mathbf{V}_{j*}\right)$.

First, since $\mathbf{V}_j'\boldsymbol{\Sigma}\mathbf{V}_{j*}$ is a scalar, $\mathbf{V}_j'\boldsymbol{\Sigma}\mathbf{V}_{j*} = tr\left(\mathbf{V}_j'\boldsymbol{\Sigma}\mathbf{V}_{j*}\right) = tr\left(\mathbf{V}_j\mathbf{V}'_{j*}\boldsymbol{\Sigma}\right)$. Then,

$E\left(\mathbf{V}_j'\boldsymbol{\Sigma}\mathbf{V}_{j*}\right) = tr\left[E\left(\mathbf{V}_j\mathbf{V}'_{j*}\right)\boldsymbol{\Sigma}\right]$. Using equations (3) and (4), we simplify

When $j = j*$

$$\begin{aligned}E_V\left(\mathbf{V}_j'\boldsymbol{\Sigma}\mathbf{V}_{j*}\right) &= tr\left(\frac{1}{A}\mathbf{I}_A\boldsymbol{\Sigma}\right) \\ &= \frac{tr(\boldsymbol{\Sigma})}{A} \\ &= \frac{1}{A-1}\left[tr(\boldsymbol{\Sigma}) - \frac{tr(\boldsymbol{\Sigma})}{A}\right]\end{aligned}$$

When $j \neq j*$.

$$E_V\left(\mathbf{V}_j{}'\boldsymbol{\Sigma}\mathbf{V}_{j*}\right)=tr\left[\frac{1}{A(A-1)}(\mathbf{J}_A-\mathbf{I}_A)\boldsymbol{\Sigma}\right]$$
$$=\frac{1}{A-1}\frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}{A}+\frac{1}{A-1}\left[-\frac{tr(\boldsymbol{\Sigma})}{A}\right]$$

As a result,

$$E_V\left(\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}'\right)=\frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}{A(A-1)}(\mathbf{J}_A-\mathbf{I}_A)+\frac{1}{A-1}\mathbf{P}_A\left[tr(\boldsymbol{\Sigma})\right] \qquad (9)$$

Note that $\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}=\frac{1}{N-1}\mathbf{1}'\mathbf{y}'\mathbf{P}_N\mathbf{y}\mathbf{1}$ and $\mathbf{y}\mathbf{1}=\left((A\mu_s)\right)=\left((A\mu+A\beta_s)\right)$, $s=1,\ldots N$. Define $\sigma_S^2=\frac{1}{N-1}\boldsymbol{\beta}'\mathbf{P}_N\boldsymbol{\beta}$. As a result,

$$\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}=\frac{1}{N-1}\left((A\mu+A\beta_s)\right)'\mathbf{P}_N\left((A\mu+A\beta_s)\right)$$
$$=\frac{A^2}{N-1}\boldsymbol{\beta}'\mathbf{P}_N\boldsymbol{\beta}$$
$$=A^2\sigma_S^2$$

Substitute this result, then

$$\begin{aligned}E_V\left(\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}'\right)\otimes\mathbf{P}_N&=\left(\frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}{A(A-1)}(\mathbf{J}_A-\mathbf{I}_A)+\frac{1}{A-1}\mathbf{P}_A\left[tr(\boldsymbol{\Sigma})\right]\right)\otimes\mathbf{P}_N\\&=\frac{A\sigma_S^2}{A-1}(\mathbf{J}_A-\mathbf{I}_A)\otimes\mathbf{P}_N+\frac{A\bar{\sigma}^2}{A-1}\mathbf{P}_A\otimes\mathbf{P}_N\end{aligned} \qquad (10)$$

Sum over (8) and (10):

$$\begin{aligned}Var_{UV}\left[vec(\mathbf{Y})\right]&=E_V\left(Var_{U|V}\left[vec(\mathbf{Y})\right]\right)+Var_V\left(E_{U|V}\left[vec(\mathbf{Y})\right]\right)\\&=\frac{A\sigma_S^2}{A-1}(\mathbf{J}_A-\mathbf{I}_A)\otimes\mathbf{P}_N+\frac{A\bar{\sigma}^2}{A-1}\mathbf{P}_A\otimes\mathbf{P}_N+\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\end{aligned} \qquad (11)$$

Simplifying the Model Assuming no Interaction:

In order to simplify the problem, we first assume $\varepsilon_{sa}=0$ (No interaction between effect of units and effect of treatments. This implies the treatment effect will not change with respect to different unit and the unit effect will not change with respect to different treatments). With this assumption we simplify the model (1) as $y_{sa}=\mu+\beta_s+\alpha_a$.

As a result, $\sigma_{aa*}=\frac{1}{N-1}\sum_{s=1}^{N}(y_{sa}-\mu_a)(y_{sa*}-\mu_{a*})=\frac{1}{N-1}\sum_{s=1}^{N}\beta_s^2=\sigma_S^2$. $\boldsymbol{\Sigma}$ can be simplified as $\boldsymbol{\Sigma}=\sigma_S^2\mathbf{J}_A$ and $\bar{\sigma}^2=\sigma_S^2$. Substitute $\bar{\sigma}^2=\sigma_S^2$ to equation(11), then

$$Var_{UV}\left[vec(\mathbf{Y})\right]=\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N+\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N$$

Re-arranging Terms into the Sample and Remainder

We re-arrange and collapse the random variables into a set corresponding to the weighted sample totals, $\mathbf{Y}_I^*$, and a set corresponding to the weighted remaining totals for each treatment, $\mathbf{Y}_{II}^*$ such that $\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}\mathbf{Y}^*$ where $\mathbf{Y}^*=vec(\mathbf{Y})$.

We assume equal sample size for each treatment. Let

$$\mathbf{K}_I=\frac{1}{N}\bigoplus_{j=1}^{A}\left(\delta_{j1}\mathbf{1}_n'\quad \delta_{j2}\mathbf{1}_n'\quad\cdots\quad \delta_{jA}\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}}\right),$$

where δ_{jj*} is an indicator variable with value one when $j=j*$ and zero otherwise. $\mathbf{K}_I$ can be written as

$$\mathbf{K}_I=\frac{1}{N}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}}\right)$$

where $\boldsymbol{\delta}_j=\left(\delta_{j1}\quad\delta_{j2}\quad\cdots\quad\delta_{jA}\right)'$ is a vector of length $A\times 1$ with a value of one in row j and zero otherwise. Note that $\mathbf{Y}^*=vec(\mathbf{Y})=\left(\mathbf{Y}_1'\quad\mathbf{Y}_2'\quad\cdots\quad\mathbf{Y}_A'\right)'$ where $\mathbf{Y}_j=\left(Y_{1j}\quad Y_{2j}\quad\ldots\quad Y_{(N-1)j}\quad Y_{Nj}\right)'$ so that

$$\mathbf{K}_I\mathbf{Y}^*=\frac{1}{N}\left[\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\right]\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\\ \vdots\\ \mathbf{Y}_A\end{pmatrix}$$

$$=\frac{1}{N}\begin{pmatrix}\left(\left(\mathbf{1}_n'\quad\mathbf{0}_n'\quad\cdots\quad\mathbf{0}_n'\right)\;\vdots\;\underset{1\times(N-nA)}{\mathbf{0}}\right)\mathbf{Y}_1\\ \left(\left(\mathbf{0}_n'\quad\mathbf{1}_n'\quad\cdots\quad\mathbf{0}_n'\right)\;\vdots\;\underset{1\times(N-nA)}{\mathbf{0}}\right)\mathbf{Y}_2\\ \vdots\\ \left(\left(\mathbf{0}_n'\quad\mathbf{0}_n'\quad\cdots\quad\mathbf{1}_n'\right)\;\vdots\;\underset{1\times(N-nA)}{\mathbf{0}}\right)\mathbf{Y}_A\end{pmatrix}.$$

Define $f=\frac{n}{N}$, and the sample mean for treatment j^{th} as $\bar{Y}_{jI}=\frac{1}{n}\sum_{i=j(n-1)+1}^{jn}Y_{ij}$,

$\mathbf{K}_I\mathbf{Y}^*=f\begin{pmatrix}\bar{Y}_{1I}\\ \bar{Y}_{2I}\\ \vdots\\ \bar{Y}_{AI}\end{pmatrix}$. Let us define $\bar{\mathbf{Y}}_I=\left(\bar{Y}_{1I}\quad\ldots\quad\bar{Y}_{jI}\quad\ldots\quad\bar{Y}_{AI}\right)'$, then $\mathbf{K}_I\mathbf{Y}^*=f\bar{\mathbf{Y}}_I$.

Similarly, let

$$\mathbf{K}_{II}=\frac{1}{N}\left(\begin{array}{c:c:c:c}(\mathbf{0}'_n\ \ \mathbf{1}'_n\ \cdots\ \mathbf{1}'_n)\ \vdots\ \mathbf{1}'_{N-nA} & \mathbf{0}'_N & \cdots & \mathbf{0}'_N \\ \hdashline \mathbf{0}'_N & (\mathbf{1}'_n\ \ \mathbf{0}'_n\ \cdots\ \mathbf{1}'_n)\ \vdots\ \mathbf{1}'_{N-nA} & \cdots & \mathbf{0}'_N \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \mathbf{0}'_N & \mathbf{0}'_N & \cdots & (\mathbf{1}'_n\ \ \mathbf{1}'_n\ \cdots\ \mathbf{0}'_n)\ \vdots\ \mathbf{1}'_{N-nA}\end{array}\right).$$

or $\mathbf{K}_{II}=\frac{1}{N}\left[\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A-\boldsymbol{\delta}'_j\right)\otimes\mathbf{1}'_n\ \vdots\ \mathbf{1}'_{N-nA}\right)\right]$. Then

$$\mathbf{K}_{II}\mathbf{Y}^*=\frac{1}{N}\left[\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A-\boldsymbol{\delta}'_j\right)\otimes\mathbf{1}'_n\ \vdots\ \mathbf{1}'_{N-nA}\right)\right]\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\\ \vdots\\ \mathbf{Y}_A\end{pmatrix}=\frac{1}{N}\begin{pmatrix}\sum_{i=n+1}^{N}Y_{i1}\\ \sum_{i=1}^{n}Y_{i2}+\sum_{i=2n+1}^{N}Y_{i2}\\ \vdots\\ \sum_{i=(A-1)n+1}^{N}Y_{iA}\end{pmatrix}.$$ Let us define the average response for the remaining random variables with treatment j^{th} as

$$\bar{Y}_{jII}=\frac{1}{N-n}\left(\sum_{i=1}^{N}Y_{ij}-n\bar{Y}_{jI}\right)$$

Then

$$\mathbf{K}_{II}\mathbf{Y}^*=(1-f)\begin{pmatrix}\bar{Y}_{1II}\\ \bar{Y}_{2II}\\ \vdots\\ \bar{Y}_{AII}\end{pmatrix}$$

Let us define $\bar{\mathbf{Y}}_{II}=\left(\bar{Y}_{1II}\ \ \ldots\ \ \bar{Y}_{jII}\ \ \ldots\ \ \bar{Y}_{AII}\right)'$, then $\mathbf{K}_{II}\mathbf{Y}^*=(1-f)\bar{\mathbf{Y}}_{II}$.

Summarizing these results, $\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}=\frac{1}{N}\left(\begin{array}{c}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j\otimes\mathbf{1}'_n\ \vdots\ \underset{1\times(N-nA)}{\mathbf{0}}\right)\\ \hdashline \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A-\boldsymbol{\delta}'_j\right)\otimes\mathbf{1}'_n\ \vdots\ \mathbf{1}'_{N-nA}\right)\end{array}\right)$ and $\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}f\bar{\mathbf{Y}}_I\\ (1-f)\bar{\mathbf{Y}}_{II}\end{pmatrix}$.

Determining the Expected Value

We use these expressions to form the expected value and variance of $\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}$.

Since $\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}\mathbf{Y}^*$ and $E_{UV}\left(\mathbf{Y}^*\right)=\mu\mathbf{1}_{NA}$,

$$E_{UV}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\frac{1}{N}\begin{pmatrix}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{1\times(N-nA)}\right)\\ \text{-----------}\\ \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A'-\boldsymbol{\delta}_j'\right)\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\end{pmatrix}\left(\mu\mathbf{1}_{NA}\right)$$

$$=\frac{1}{N}\begin{pmatrix}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{1\times(N-nA)}\right)\\ \text{-----------}\\ \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A'-\boldsymbol{\delta}_j'\right)\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\end{pmatrix}\left(\mu\mathbf{1}_{NA}\right).$$

$$=\frac{1}{N}\begin{pmatrix}\bigoplus_{j=1}^{A} n\mu\\ \text{-----------}\\ \bigoplus_{j=1}^{A}\left(N-n\right)\mu\end{pmatrix}\mathbf{1}_A$$

We express this as $E_{UV}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}f\bigoplus_{j=1}^{A}\mu\\ \text{-----------}\\ (1-f)\bigoplus_{j=1}^{A}\mu\end{pmatrix}\mathbf{1}_A=\left[\begin{pmatrix}f\\ 1-f\end{pmatrix}\otimes\mathbf{1}_A\right]\mu$

Determining the Partitioned Variance

Next, we consider expressions for the variance:

$$\text{var}_{UV}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}\text{var}_{UV}\left(\mathbf{Y}^*\right)\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}'$$

$$=\begin{pmatrix}\mathbf{V}_I & \mathbf{V}_{I,II}\\ \mathbf{V}_{I,II}' & \mathbf{V}_{II}\end{pmatrix}$$

$$=\begin{pmatrix}\mathbf{K}_I\,\text{var}_{UV}\left(\mathbf{Y}^*\right)\mathbf{K}_I' & \mathbf{K}_I\,\text{var}_{UV}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}'\\ \mathbf{K}_{II}\,\text{var}_{UV}\left(\mathbf{Y}^*\right)\mathbf{K}_I' & \mathbf{K}_{II}\,\text{var}_{UV}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}'\end{pmatrix}$$

Now $\begin{pmatrix}\mathbf{K}_I\\ \mathbf{K}_{II}\end{pmatrix}=\frac{1}{N}\begin{pmatrix}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{1\times(N-nA)}\right)\\ \text{-----------}\\ \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A'-\boldsymbol{\delta}_j'\right)\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\end{pmatrix}$ and $\text{var}_{UV}\left(\mathbf{Y}^*\right)=\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N+\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N$ so that

$\mathbf{V}_I=\mathbf{K}_I\,\text{var}_{UV}\left(\mathbf{Y}^*\right)\mathbf{K}_I'$ is given by

$$\mathbf{V}_I=\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N+\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$

$$=\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}+\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left[\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$

For the first item:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$

$$=\frac{\sigma_S^2}{N^2}\left(\begin{array}{c|c|c|c}
\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_2\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_A\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\ \hline
\left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_2\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_A\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\ \hline
\vdots & \vdots & \ddots & \vdots\\ \hline
\left(\boldsymbol{\delta}_A'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_A'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_2\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_A'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_A\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}
\end{array}\right)$$

Now

$$\begin{aligned}
&\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\
&=\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}-\frac{1}{N}\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}.\\
&=n-\frac{n^2}{N}=n\left(1-\frac{n}{N}\right)\\
&=n\left(1-f\right)
\end{aligned}$$

Also,

$$\begin{aligned}
&\left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\
&=\left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}-\frac{1}{N}\left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}.\\
&=n\left(-\frac{n}{N}\right)\\
&=-nf
\end{aligned}$$

As a result,

$$\begin{aligned}
&\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\
&=\frac{n}{N^2}\sigma_S^2\left(\begin{array}{c|c|c|c}
(1-f) & (-f) & \cdots & (-f)\\ \hline
(-f) & (1-f) & \cdots & (-f)\\ \hline
\vdots & \vdots & \ddots & \vdots\\ \hline
(-f) & (-f) & \cdots & (1-f)
\end{array}\right)\\
&=\frac{f}{N}\sigma_S^2\left(\mathbf{I}_A-f\mathbf{J}_A\right)
\end{aligned}$$

For the second item:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left[\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$

$$=\frac{\sigma_A^2}{N^2}\left(\begin{array}{c|c|c|c}
\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_2\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_A\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\ \hline
\left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_2\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_A\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\ \hline
\vdots & \vdots & \ddots & \vdots\\ \hline
\left(\boldsymbol{\delta}_A'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_A'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_2\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_A'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_A\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}
\end{array}\right)$$

Now

$$\begin{aligned}
&\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\
&=\left(1-\frac{1}{A}\right)\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}.\\
&=\left(1-\frac{1}{A}\right)n^2
\end{aligned}$$

Also,

$$\begin{aligned}
&\left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}\\
&=\left(-\frac{1}{A}\right)\left(\boldsymbol{\delta}_2'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\boldsymbol{\delta}_1\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}.\\
&=n^2\left(-\frac{1}{A}\right)
\end{aligned}$$

As a result,

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left[\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$

$$=\frac{\sigma_A^2}{N^2}\left(\begin{array}{c|c|c|c}
n^2\left(1-\frac{1}{A}\right) & n^2\left(-\frac{1}{A}\right) & \cdots & n^2\left(-\frac{1}{A}\right)\\ \hline
n^2\left(-\frac{1}{A}\right) & n^2\left(1-\frac{1}{A}\right) & \cdots & n^2\left(-\frac{1}{A}\right)\\ \hline
\vdots & \vdots & \ddots & \vdots\\ \hline
n^2\left(-\frac{1}{A}\right) & n^2\left(-\frac{1}{A}\right) & \cdots & n^2\left(1-\frac{1}{A}\right)
\end{array}\right)$$

$$=\frac{n^2\sigma_A^2}{N^2}\mathbf{P}_A=f^2\sigma_A^2\mathbf{P}_A$$

Adding these two items:

$$\begin{aligned}\mathbf{V}_I &= \frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2 \mathbf{J}_A \otimes \mathbf{P}_N + \sigma_A^2 \mathbf{P}_A \otimes \mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\boldsymbol{\delta}_j \otimes \mathbf{1}_n \\ \mathbf{0}_{N-nA}\end{pmatrix} \\ &= \frac{f}{N}\sigma_S^2\left(\mathbf{I}_A - f\mathbf{J}_A\right) + f^2\sigma_A^2 \mathbf{P}_A\end{aligned}$$

Similarly, $\mathbf{V}_{I,II} = \mathbf{K}_I \operatorname{var}_{UV}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}'$

$$\begin{aligned}\mathbf{V}_{I,II} &= \frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2 \mathbf{J}_A \otimes \mathbf{P}_N + \sigma_A^2 \mathbf{P}_A \otimes \mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_j\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} \\ &= \frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2 \mathbf{J}_A \otimes \mathbf{P}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_j\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} + \frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\left[\sigma_A^2 \mathbf{P}_A \otimes \mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_j\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix}\end{aligned}.$$

For the first item:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2 \mathbf{J}_A \otimes \mathbf{P}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_j\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix}$$

$$= \frac{\sigma_S^2}{N^2}\left(\begin{array}{c|c|c|c}
\left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_1\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_2\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_A\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} \\ \hline
\left(\boldsymbol{\delta}_2' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_1\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_2' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_2\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_2' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_A\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
\left(\boldsymbol{\delta}_A' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_1\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\boldsymbol{\delta}_A' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_2\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_A' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_A\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix}
\end{array}\right)$$

Now

$$\begin{aligned}&\left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_1\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} \\ &= \left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_1\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} - \frac{1}{N}\left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vert\; \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\left(\mathbf{1}_A - \boldsymbol{\delta}_1\right) \otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA}\end{pmatrix} \\ &= 0 - \frac{1}{N}\left(n\left(N-n\right)\right) \\ &= -n\left(1-f\right)\end{aligned}$$

While,

$$\begin{aligned}
&\left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\mathbf{P}_N \begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_2\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} \\
&= \left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_2\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} - \frac{1}{N}\left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N' \begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_2\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} \\
&= n - \frac{1}{N}\left(n\left(N-n\right)\right) \\
&= n - n\left(1-f\right) \\
&= nf
\end{aligned}
\quad .$$

Hence:

$$\begin{aligned}
&\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left[\sigma_S^2 \mathbf{J}_A \otimes \mathbf{P}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_j\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} \\
&= \frac{\sigma_S^2}{N^2}\left(\begin{array}{c:c:c:c} -n\left(1-f\right) & nf & \cdots & nf \\ \hdashline nf & -n\left(1-f\right) & \cdots & nf \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline nf & nf & \cdots & -n\left(1-f\right) \end{array}\right) \\
&= f\,\frac{\sigma_S^2}{N}\left(\begin{array}{c:c:c:c} f-1 & f & \cdots & f \\ \hdashline f & f-1 & \cdots & f \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline f & f & \cdots & f-1 \end{array}\right) \\
&= f\,\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A - \mathbf{I}_A\right]
\end{aligned}$$

For the second item:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left[\sigma_A^2 \mathbf{P}_A \otimes \mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_j\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix}$$

$$== \frac{\sigma_A^2}{N^2}\left(\begin{array}{c:c:c:c}
\left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_1\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} & \left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_2\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_1' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_A\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} \\ \hdashline
\left(\boldsymbol{\delta}_2' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_1\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} & \left(\boldsymbol{\delta}_2' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_2\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_2' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_A\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} \\ \hdashline
\vdots & \vdots & \ddots & \vdots \\ \hdashline
\left(\boldsymbol{\delta}_A' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_1\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} & \left(\boldsymbol{\delta}_A' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_2\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix} & \cdots & \left(\boldsymbol{\delta}_A' \otimes \mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix} \left(\mathbf{1}_A - \boldsymbol{\delta}_A\right)\otimes \mathbf{1}_n \\ \mathbf{1}_{N-nA} \end{pmatrix}
\end{array}\right)$$

Now

$$\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$
$$=\left(1-\frac{1}{A}\right)\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$
$$=\left(1-\frac{1}{A}\right)n\left(N-n\right)$$

while

$$\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$
$$=\left(-\frac{1}{A}\right)\left(\boldsymbol{\delta}_1'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$
$$=\left(-\frac{1}{A}\right)n\left(N-n\right)$$

Hence:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \mid \mathbf{0}_{N-nA}'\right)\left[\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right]\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$
$$=\frac{\sigma_A^2}{N^2}\left(\begin{array}{c:c:c:c}\left(1-\frac{1}{A}\right)n\left(N-n\right) & \left(-\frac{1}{A}\right)n\left(N-n\right) & \cdots & \left(-\frac{1}{A}\right)n\left(N-n\right)\\ \hdashline \left(-\frac{1}{A}\right)n\left(N-n\right) & \left(1-\frac{1}{A}\right)n\left(N-n\right) & \cdots & \left(-\frac{1}{A}\right)n\left(N-n\right)\\ \hdashline \vdots & \vdots & \ddots & \vdots\\ \hdashline \left(-\frac{1}{A}\right)n\left(N-n\right) & \left(-\frac{1}{A}\right)n\left(N-n\right) & \cdots & \left(1-\frac{1}{A}\right)n\left(N-n\right)\end{array}\right)$$
$$=f\left(1-f\right)\sigma_A^2\left(\begin{array}{c:c:c:c}\left(1-\frac{1}{A}\right) & \left(-\frac{1}{A}\right) & \cdots & \left(-\frac{1}{A}\right)\\ \hdashline \left(-\frac{1}{A}\right) & \left(1-\frac{1}{A}\right) & \cdots & \left(-\frac{1}{A}\right)\\ \hdashline \vdots & \vdots & \ddots & \vdots\\ \hdashline \left(-\frac{1}{A}\right) & \left(-\frac{1}{A}\right) & \cdots & \left(1-\frac{1}{A}\right)\end{array}\right)$$
$$=f\left(1-f\right)\sigma_A^2\mathbf{P}_A$$

Hence, $\mathbf{V}_{I,II}=f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A-\mathbf{I}_A\right]+f\left(1-f\right)\sigma_A^2\mathbf{P}_A$.

$$
\begin{aligned}
\mathbf{V}_{II} &= \mathbf{K}_{II}\ \mathrm{var}_{UV}\left(\mathbf{Y}^*\right)\mathbf{K}'_{II} = \mathbf{K}_{II}\left(\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N+\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right)\mathbf{K}'_{II}\\
&= \frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\left(\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N+\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\
&= \frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\left(\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}+\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\left(\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}
\end{aligned}
$$

For the first item:

$$
\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\left(\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}
$$

$$
=\frac{\sigma_S^2}{N^2}\left(\begin{array}{c|c|c|c}
\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\ \hline
\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)'_2\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\ \hline
\vdots & \vdots & \ddots & \vdots\\ \hline
\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}
\end{array}\right)
$$

Now

$$
\begin{aligned}
&\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\
&=\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}-\frac{1}{N}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{1}_N\mathbf{1}'_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\
&=\left(N-n\right)-\frac{1}{N}\left(\left(N-n\right)\left(N-n\right)\right)\\
&=\left(N-n\right)\left(1-\frac{1}{N}\left(N-n\right)\right)\\
&= f\left(N-n\right)\\
&= n-nf
\end{aligned}
$$

While

$$
\begin{aligned}
&\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{P}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\
&=\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}-\frac{1}{N}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\mathbf{1}_N\mathbf{1}'_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}.\\
&=\left(N-2n\right)-\frac{1}{N}\left(N-n\right)\left(N-n\right)\\
&=\left(N-2n\right)-\left(1-f\right)\left(N-n\right)\\
&=N-2n-\left(N-n-fN+fn\right)\\
&=-fn
\end{aligned}
$$

Hence:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\left(\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$

$$=\frac{\sigma_S^2}{N^2}\left(\begin{array}{c|c|c|c} n-nf & -nf & \cdots & -nf\\ \hline -nf & n-nf & \cdots & -nf\\ \hline \vdots & \vdots & \ddots & \vdots\\ \hline -nf & -nf & \cdots & n-nf\end{array}\right)$$

$$=f\,\frac{\sigma_S^2}{N}\left(\begin{array}{c|c|c|c} 1-f & -f & \cdots & -f\\ \hline -f & 1-f & \cdots & -f\\ \hline \vdots & \vdots & \ddots & \vdots\\ \hline -f & -f & \cdots & 1-f\end{array}\right)$$

$$=f\,\frac{\sigma_S^2}{N}\left[\mathbf{I}_A-f\mathbf{J}_A\right]$$

For the second item:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\left(\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$

$$=\frac{\sigma_A^2}{N^2}\left(\begin{array}{c|c|c|c}
\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\ \hline
\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)'_2\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}\\ \hline
\vdots & \vdots & \ddots & \vdots\\ \hline
\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix} & \cdots & \left(\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)'\otimes\mathbf{1}_n' \vdots \mathbf{1}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_A\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}
\end{array}\right)$$

Now

$$\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\left(1-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$

$$=\left(1-\frac{1}{A}\right)\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$

$$=\left(1-\frac{1}{A}\right)\left(N-n\right)^2$$

While

$$\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\left(-\frac{1}{A}\right)\mathbf{J}_N\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$
$$=\left(-\frac{1}{A}\right)\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_1\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\mathbf{1}_N\mathbf{1}_N'\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_2\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}.$$
$$=\left(-\frac{1}{A}\right)\left(N-n\right)^2$$

Hence:

$$\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{1}_{N-nA}'\right)\left(\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{1}_{N-nA}\end{pmatrix}$$
$$=\frac{\sigma_A^2}{N^2}\begin{pmatrix}\left(1-\frac{1}{A}\right)(N-n)^2 & \left(-\frac{1}{A}\right)(N-n)^2 & \cdots & \left(-\frac{1}{A}\right)(N-n)^2\\ \left(-\frac{1}{A}\right)(N-n)^2 & \left(1-\frac{1}{A}\right)(N-n)^2 & \cdots & \left(-\frac{1}{A}\right)(N-n)^2\\ \vdots & \vdots & \ddots & \vdots\\ \left(-\frac{1}{A}\right)(N-n)^2 & \left(-\frac{1}{A}\right)(N-n)^2 & \cdots & \left(1-\frac{1}{A}\right)(N-n)^2\end{pmatrix}$$
$$=\frac{\sigma_A^2}{N^2}\left(N-n\right)^2\mathbf{P}_A$$
$$=\left(1-f\right)^2\sigma_A^2\mathbf{P}_A$$

Adding the two items:

$$\mathbf{V}_{II}=\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$
$$+\frac{1}{N^2}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)'\otimes\mathbf{1}_n' \;\vdots\; \mathbf{0}_{N-nA}'\right)\left(\sigma_A^2\mathbf{P}_A\otimes\mathbf{J}_N\right)\bigoplus_{j=1}^{A}\begin{pmatrix}\left(\mathbf{1}_A-\boldsymbol{\delta}_j\right)\otimes\mathbf{1}_n\\ \mathbf{0}_{N-nA}\end{pmatrix}$$
$$=f\frac{\sigma_S^2}{N}\left[\mathbf{I}_A-f\mathbf{J}_A\right]+\left(1-f\right)^2\sigma_A^2\mathbf{P}_A$$

In summary, $\text{var}_{UV}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}\mathbf{V}_I & \mathbf{V}_{I,II}\\ \mathbf{V}_{I,II}' & \mathbf{V}_{II}\end{pmatrix}$, where

$$\mathbf{V}_I=\frac{f}{N}\sigma_S^2\left(\mathbf{I}_A-f\mathbf{J}_A\right)+f^2\sigma_A^2\mathbf{P}_A$$
$$\mathbf{V}_{I,II}=f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A-\mathbf{I}_A\right]+f\left(1-f\right)\sigma_A^2\mathbf{P}_A$$
$$\mathbf{V}_{II}=f\frac{\sigma_S^2}{N}\left[\mathbf{I}_A-f\mathbf{J}_A\right]+\left(1-f\right)^2\sigma_A^2\mathbf{P}_A.$$

Predicting Parameters corresponding to Combinations of Treatment Means

We are interested in a linear combination of treatment means given by $P = \mathbf{g}'\mathbf{Y}^*$. In terms of the partitioned vector $\begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I \\ \mathbf{K}_{II} \end{pmatrix} \mathbf{Y}^*$, the target parameter is given by

$P_j = \begin{pmatrix} \mathbf{e}'_{jI} & \mathbf{e}'_{jII} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \mathbf{e}'_{jI}\mathbf{Y}_I^* + \mathbf{e}'_{jII}\mathbf{Y}_{II}^*$ where all elements of $\mathbf{e}_{jI}$ and $\mathbf{e}_{jII}$ are equal to zero except the element in row j which is equal to one. Define $\mathbf{e}'_{jI} = \mathbf{e}'_{jII} = \mathbf{e}'_j$. Since $\begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \begin{pmatrix} f\bar{\mathbf{Y}}_I \\ (1-f)\bar{\mathbf{Y}}_{II} \end{pmatrix}$,

then

$$\begin{aligned}
P_j &= \begin{pmatrix} \mathbf{e}'_{jI} & \mathbf{e}'_{jII} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} \\
&= \mathbf{e}'_j\mathbf{Y}_I^* + \mathbf{e}'_j\mathbf{Y}_{II}^* \\
&= f\bar{Y}_{jI} + (1-f)\bar{Y}_{jII} \\
&= \frac{f}{n}\sum_{i=j(n-1)+1}^{jn} Y_{ij} + \frac{1-f}{N-n}\left(\sum_{i=1}^{N} Y_{ij} - n\bar{Y}_{jI}\right) \\
&= \frac{1}{N}\sum_{i=j(n-1)+1}^{jn} Y_{ij} + \frac{1}{N}\left(\sum_{i=1}^{N} Y_{ij} - n\bar{Y}_{jI}\right) \\
&= \frac{1}{N}\sum_{i=1}^{N} Y_{ij}
\end{aligned}$$

Where P_j indicates the mean of the j^{th} postion in the permutation of treatments.

We require the predictor to be a linear function of the random variables in the sample, to be unbiased, and to minimize the expected value of the mean squared error. Collection of the study data will result in realizing the values of $\mathbf{Y}_I^*$.

Refer to Argetina2006-lec1a.doc which gives the general method to find the BLUP.

For our special sample:

Target: $P_j = \frac{1}{N}\sum_{i=1}^{N} Y_{ij}$

Collapsing: $\begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \begin{pmatrix} f\bar{\mathbf{Y}}_I \\ (1-f)\bar{\mathbf{Y}}_{II} \end{pmatrix}$

We know $E_{UV}\begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \left[\begin{pmatrix} f \\ 1-f \end{pmatrix} \otimes \mathbf{1}_A\right]\mu$ and $\begin{pmatrix} \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}'_{I,II} & \mathbf{V}_{II} \end{pmatrix}$

Now $\begin{pmatrix} \mathbf{g}'_I & \mathbf{g}'_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_j & \mathbf{e}'_j \end{pmatrix}$, $\mathbf{X}_I = f\mathbf{1}_A$ (12) and $\mathbf{X}_{II} = (1-f)\mathbf{1}_A$ (13).

To evaluate the predictor, we find note $\mathbf{V}_I^{-1}$ first.

Since we know:

$$\begin{aligned}\mathbf{V}_I &= \frac{f}{N}\sigma_S^2\left(\mathbf{I}_A - f\mathbf{J}_A\right) + f^2\sigma_A^2\mathbf{P}_A \\ &= \frac{f}{N}\sigma_S^2\left(\mathbf{I}_A - f\mathbf{J}_A\right) + f^2\sigma_A^2\left(\mathbf{I}_A - \frac{1}{A}\mathbf{J}_A\right) \\ &= \left(\frac{f}{N}\sigma_S^2 + f^2\sigma_A^2\right)\mathbf{I}_A - \left(\frac{f^2\sigma_A^2}{A} + \frac{f^2\sigma_S^2}{N}\right)\mathbf{J}_A\end{aligned}$$

Define $a = \left(\frac{f}{N}\sigma_S^2 + f^2\sigma_A^2\right)$ and $b = \left(\frac{f^2\sigma_A^2}{A} + \frac{f^2\sigma_S^2}{N}\right)$, plug in $\mathbf{V}_I$, then

$$\begin{aligned}\mathbf{V}_I &= a\mathbf{I}_A - b\mathbf{J}_A \\ &= a\left(\mathbf{I}_A - \frac{b}{a}\mathbf{J}_A\right)\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{V}_I^{-1} &= a^{-1}\left(\mathbf{I}_A + \frac{1}{\frac{a}{b} - A}\mathbf{J}_A\right) \qquad (14) \\ &= \frac{1}{a}\mathbf{I}_A + \frac{b}{a^2 - Aab}\mathbf{J}_A\end{aligned}$$

Other terms simplify:
Using (12) and (14),

$$\begin{aligned}\hat{\alpha} &= \left(\mathbf{X}_I{}'\mathbf{V}_I^{-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I{}'\mathbf{V}_I^{-1}\mathbf{Y}_I^* \\ &= \left[f\mathbf{1}_A'\left(\frac{1}{a}\mathbf{I}_A + \frac{b}{a^2 - Aab}\mathbf{J}_A\right)f\mathbf{1}_A\right]^{-1}\left(f\mathbf{1}_A'\right)\left(\frac{1}{a}\mathbf{I}_A + \frac{b}{a^2 - Aab}\mathbf{J}_A\right)f\bar{\mathbf{Y}}_I \\ &= \left(\frac{Af^2}{a - Ab}\right)^{-1}\frac{f^2}{a - Ab}\mathbf{1}_A'\bar{\mathbf{Y}}_I \qquad (15) \\ &= \frac{1}{A}\mathbf{1}_A'\bar{\mathbf{Y}}_I \\ &= \bar{\bar{Y}}_I\end{aligned}$$

where $\bar{\bar{Y}}_I$ is the overall sample mean.

Next we simplify the term $\mathbf{V}_{II,I}\mathbf{V}_I^{-1}$. We know:

$$\begin{aligned}\mathbf{V}_{I,II} &= f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A-\mathbf{I}_A\right]+f\left(1-f\right)\sigma_A^2\mathbf{P}_A\\ &= f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A-\mathbf{I}_A\right]+f\left(1-f\right)\sigma_A^2\left(\mathbf{I}_A-\frac{1}{A}\mathbf{J}_A\right)\\ &=\left(f\left(1-f\right)\sigma_A^2-f\frac{\sigma_S^2}{N}\right)\mathbf{I}_A+\left(\frac{\sigma_S^2 f^2}{N}-\frac{f\left(1-f\right)\sigma_A^2}{A}\right)\mathbf{J}_A\end{aligned}$$

Define $c=f\left(1-f\right)\sigma_A^2-f\frac{\sigma_S^2}{N}$ and $d=\frac{\sigma_S^2 f^2}{N}-\frac{f\left(1-f\right)\sigma_A^2}{A}$

Hence, $\mathbf{V}_{I,II}=c\mathbf{I}_A+d\mathbf{J}_A$

Then

$$\begin{aligned}\mathbf{V}_{II,I}\mathbf{V}_I^{-1}=\mathbf{V}_{I,II}'\mathbf{V}_I^{-1} &=\left(c\mathbf{I}_A+d\mathbf{J}_A\right)'\left(\frac{1}{a}\mathbf{I}_A+\frac{b}{a^2-Aab}\mathbf{J}_A\right)\\ &=\left(\frac{c}{a}\mathbf{I}_A+\frac{bc}{a^2-Aab}\mathbf{J}_A+\frac{d}{a}\mathbf{J}_A+\frac{bdA}{a^2-Aab}\mathbf{J}_A\right)\\ &=\left[\frac{c}{a}\mathbf{I}_A+\left(\frac{bc+d(a-Ab)+bdA}{a^2-Aab}\right)\mathbf{J}_A\right]\\ &=\left[\frac{c}{a}\mathbf{I}_A+\left(\frac{bc+ad}{a^2-Aab}\right)\mathbf{J}_A\right]\end{aligned} \qquad (16).$$

Using (13), (15) and (16),

$$\begin{aligned}\hat{P}_j &= \mathbf{g}_I'\mathbf{Y}_I^*+\mathbf{g}_{II}'\left[\mathbf{X}_{II}\hat{\alpha}+\mathbf{V}_{II,I}\mathbf{V}_I^{-1}\left(\mathbf{Y}_I^*-\mathbf{X}_I\hat{\alpha}\right)\right]\\ &=\mathbf{e}_j'\mathbf{Y}_I^*+\mathbf{e}_j'\left[\left(1-f\right)\bar{\bar{Y}}_I\mathbf{1}_A+\left[\frac{c}{a}\mathbf{I}_A+\left(\frac{bc+ad}{a^2-Aab}\right)\mathbf{J}_A\right]\left(\mathbf{Y}_I^*-f\bar{\bar{Y}}_I\mathbf{1}_A\right)\right]\\ &= f\bar{Y}_{jI}+\mathbf{e}_j'\left[\left(1-f\right)\bar{\bar{Y}}_I\mathbf{1}_A+\left[\frac{c}{a}\mathbf{I}_A+\left(\frac{bc+ad}{a^2-Aab}\right)\mathbf{J}_A\right]\left(f\bar{\mathbf{Y}}_I-f\bar{\bar{Y}}_I\mathbf{1}_A\right)\right]\\ &= f\bar{Y}_{jI}+\left(1-f\right)\bar{\bar{Y}}_I+\left[\frac{c}{a}\mathbf{e}_j'+\left(\frac{bc+ad}{a^2-Aab}\right)\mathbf{1}_A'\right]\left(f\bar{\mathbf{Y}}_I-f\bar{\bar{Y}}_I\mathbf{1}_A\right)\\ &= f\bar{Y}_{jI}+\left(1-f\right)\bar{\bar{Y}}_I+\left(\frac{c}{a}f\bar{Y}_{jI}-\frac{c}{a}f\bar{\bar{Y}}_I\right)+\left(\frac{bc+ad}{a^2-Aab}\right)\left(fA\bar{\bar{Y}}_I-fA\bar{\bar{Y}}_I\right)\\ &= f\bar{Y}_{jI}+\left(1-f\right)\left[\bar{\bar{Y}}_I+\frac{c}{a}\frac{f}{1-f}\left(\bar{Y}_{jI}-\bar{\bar{Y}}_I\right)\right]\end{aligned}$$

Since $c=f\left(1-f\right)\sigma_A^2-f\frac{\sigma_S^2}{N}$ and $a=\left(\frac{f}{N}\sigma_S^2+f^2\sigma_A^2\right)$, then

$$\frac{c}{a}=\frac{f\left(1-f\right)\sigma_A^2-f\frac{\sigma_S^2}{N}}{\frac{f}{N}\sigma_S^2+f^2\sigma_A^2}=\frac{N\left(1-f\right)\sigma_A^2-\sigma_S^2}{\sigma_S^2+Nf\sigma_A^2}=\frac{\left(N-n\right)\sigma_A^2-\sigma_S^2}{\sigma_S^2+n\sigma_A^2}$$

Define

$$k^* = \frac{f}{1-f}\frac{c}{a}$$
$$= \frac{f}{1-f}\frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}$$
$$= \frac{f}{1-f}\frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}$$

Thus, the estimate can be simplified as

$$\hat{P}_j = f\bar{Y}_{jI} + (1-f)\left(\bar{\bar{Y}}_I + k^*\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right)\right) \text{ where } k^* = \frac{f}{1-f}\frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}$$

Comparing the MSE between RP model and ANOVA??

Express the sample mean of j^{th} treatment $\left(\bar{Y}_{jI}\right)$ and overall sample mean $\left(\bar{\bar{Y}}_I\right)$ in terms of $\mathbf{Y}^* = vec(\mathbf{Y})$.

$$\hat{P}_j = f\bar{Y}_{jI} + (1-f)\left(\bar{\bar{Y}}_I + k^*\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right)\right)$$

$$\bar{Y}_{jI} = \frac{1}{n}\mathbf{e}_j' \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*$$

$$\bar{\bar{Y}}_I = \frac{1}{An}\mathbf{1}_A' \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*$$
$$= \frac{\mathbf{e}'\mathbf{J}_A}{An} \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*$$

Then

$$k^*\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right) = k^*\left(\frac{1}{n}\mathbf{e}_j' \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{\mathbf{e}'\mathbf{J}_A}{An} \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\right)\mathbf{Y}^*$$
$$= \frac{1}{n}k^*\mathbf{e}'\left(\mathbf{I}_A - \frac{\mathbf{J}_A}{A}\right)_j \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*$$
$$= \frac{1}{n}k^*\mathbf{e}'\mathbf{P}_A \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*$$

$$\hat{P}_j = f\frac{1}{n}\mathbf{e}_j' \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) + (1-f)\left(\frac{\mathbf{e}'\mathbf{J}_A}{An} \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^* + \frac{1}{n}k^*\mathbf{e}'\mathbf{P}_A \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*\right)$$
$$= \mathbf{e}_j'\frac{1}{n}\left[f\mathbf{I}_A + (1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right)\right] \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j' \otimes \mathbf{1}_n' \;\middle|\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*$$

Express the population mean for the j^{th} treatment in terms of $\mathbf{Y}^* = vec(\mathbf{Y})$.

$$P_j = \mathbf{e}'_j \frac{1}{N}\left(\bigoplus_{j=1}^{A} \mathbf{1}'_N\right)\mathbf{Y}^*$$
$$= \mathbf{e}'_j \frac{1}{N}\left(\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) + \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right)\mathbf{Y}^*$$

Evaluate $\hat{P}_j - P_j$

$$\hat{P}_j - P_j = \mathbf{e}'_j \frac{1}{n}\left[f\mathbf{I}_A + (1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right)\right]\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^*$$
$$-\mathbf{e}'_j \frac{1}{N}\left(\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) + \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right)\mathbf{Y}^*$$
$$= \mathbf{e}'_j\left(\frac{1}{n}\left[f\mathbf{I}_A + (1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right)\right]\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\left(\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) + \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right)\right)\mathbf{Y}^*$$
$$= \mathbf{e}'_j\left(\left[\frac{1}{N}\mathbf{I}_A + \frac{1}{n}(1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right)\right]\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right)\mathbf{Y}^*$$
$$= \mathbf{e}'_j\left(\frac{1}{n}(1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right)\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right)\mathbf{Y}^*$$

Define $\mathbf{C} = \frac{1}{n}(1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right)\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)$ and refer to $\begin{pmatrix}\mathbf{K}_I \\ \mathbf{K}_{II}\end{pmatrix} = \frac{1}{N}\begin{pmatrix}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}}\right) \\ \hdashline \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \mathbf{1}'_{N-nA}\right)\end{pmatrix}$, then $\mathbf{C} = \frac{N}{n}(1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right)\mathbf{K}_I - \mathbf{K}_{II}$. Define

$\mathbf{G} = \frac{N}{n}(1-f)\left(\frac{\mathbf{J}_A}{A} + k^*\mathbf{P}_A\right) = \frac{N}{n}(1-f)\left(-\left(\mathbf{I}_A - \frac{\mathbf{J}_A}{A}\right) + k^*\mathbf{P}_A + \mathbf{I}_A\right) = \frac{N}{n}(1-f)\left(\mathbf{I}_A + \left(k^*-1\right)\mathbf{P}_A\right)$,

then

$$\text{var}\left(\hat{P}_j - P_j\right) = \text{var}\left(\mathbf{e}'_j\mathbf{C}\mathbf{Y}^*\right)$$
$$= \mathbf{e}'_j\mathbf{C}\,\text{var}\left(\mathbf{Y}^*\right)\mathbf{C}'\mathbf{e}_j$$
$$= \mathbf{e}'_j\left(\mathbf{G}\mathbf{K}_I - \mathbf{K}_{II}\right)\text{var}\left(\mathbf{Y}^*\right)\left(\mathbf{G}\mathbf{K}_I - \mathbf{K}_{II}\right)'\mathbf{e}_j$$
$$= \mathbf{e}'_j\left(\mathbf{G}\mathbf{K}_I\text{var}\left(\mathbf{Y}^*\right)\mathbf{K}'_I\mathbf{G}' - \mathbf{K}_{II}\text{var}\left(\mathbf{Y}^*\right)\mathbf{K}'_I\mathbf{G}' - \mathbf{G}\mathbf{K}_I\text{var}\left(\mathbf{Y}^*\right)\mathbf{K}'_{II} + \mathbf{K}_{II}\text{var}\left(\mathbf{Y}^*\right)\mathbf{K}'_{II}\right)\mathbf{e}_j$$
$$= \mathbf{e}'_j\left(\mathbf{G}\mathbf{V}_I\mathbf{G}' - \mathbf{V}'_{II,II}\mathbf{G}' - \mathbf{G}\mathbf{V}_{I,II} + \mathbf{V}_{II}\right)\mathbf{e}_j$$
$$= \mathbf{e}'_j\left(\mathbf{G}\mathbf{V}_I\mathbf{G}' - 2\mathbf{G}\mathbf{V}_{I,II} + \mathbf{V}_{II}\right)\mathbf{e}_j$$

For $\mathbf{G}\mathbf{V}_I\mathbf{G}'$

$$\mathbf{G}\mathbf{V}_I\mathbf{G}' = \left[\frac{N}{n}(1-f)\left(\frac{\mathbf{J}_A}{A}+k^*\mathbf{P}_A\right)\right]\left[\frac{f}{N}\sigma_S^2\left(\mathbf{I}_A - f\mathbf{J}_A\right)+f^2\sigma_A^2\mathbf{P}_A\right]\left[\frac{N}{n}(1-f)\left(\frac{\mathbf{J}_A}{A}+k^*\mathbf{P}_A\right)\right]'$$

$$= N^2\left(\frac{1}{n}-\frac{1}{N}\right)^2\left(\frac{\mathbf{J}_A}{A}+k^*\mathbf{P}_A\right)\left[\frac{f}{N}\sigma_S^2\mathbf{I}_A - \frac{f^2}{N}\sigma_S^2\mathbf{J}_A + f^2\sigma_A^2\mathbf{P}_A\right]\left(\frac{\mathbf{J}_A}{A}+k^*\mathbf{P}_A\right)$$

$$= N^2\left(\frac{1}{n}-\frac{1}{N}\right)^2\left[\left(\frac{f}{NA}\sigma_S^2 - \frac{f^2}{N}\sigma_S^2\right)\mathbf{J}_A + \left(\frac{f}{N}\sigma_S^2 k^* + f^2\sigma_A^2 k^*\right)\mathbf{P}_A\right]\left(\frac{\mathbf{J}_A}{A}+k^*\mathbf{P}_A\right)$$

$$= N^2\left(\frac{1}{n}-\frac{1}{N}\right)^2\left[\left(\frac{f}{NA}\sigma_S^2 - \frac{f^2}{N}\sigma_S^2\right)\mathbf{J}_A + \left(\frac{f}{N}\sigma_S^2 k^* + f^2\sigma_A^2 k^*\right)k^*\mathbf{P}_A\right]$$

$$\mathbf{e}_j'\left(\mathbf{G}\mathbf{V}_I\mathbf{G}'\right)\mathbf{e}_j = \mathbf{e}_j'\left(N^2\left(\frac{1}{n}-\frac{1}{N}\right)^2\left[\left(\frac{f}{NA}\sigma_S^2 - \frac{f^2}{N}\sigma_S^2\right)\mathbf{J}_A + \left(\frac{f}{N}\sigma_S^2 k^* + f^2\sigma_A^2 k^*\right)k^*\mathbf{P}_A\right]\right)\mathbf{e}_j$$

$$= N^2\left(\frac{1}{n}-\frac{1}{N}\right)^2\left[\left(\frac{f}{NA}\sigma_S^2 - \frac{f^2}{N}\sigma_S^2\right) + \left(\frac{f}{N}\sigma_S^2 k^* + f^2\sigma_A^2 k^*\right)k^*\left(1-\frac{1}{A}\right)\right]$$

$$\mathbf{G}\mathbf{V}_{I,II} = \frac{N}{n}(1-f)\left(\frac{\mathbf{J}_A}{A}+k^*\mathbf{P}_A\right)\left(f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A - \mathbf{I}_A\right] + f(1-f)\sigma_A^2\mathbf{P}_A\right)$$

$$= \frac{N}{n}(1-f)\left(\frac{\mathbf{J}_A}{A}+k^*\mathbf{P}_A\right)\left(f^2\frac{\sigma_S^2}{N}\mathbf{J}_A - f\frac{\sigma_S^2}{N}\mathbf{I}_A + f(1-f)\sigma_A^2\mathbf{P}_A\right)$$

$$= \frac{N}{n}(1-f)\left(f^2\frac{\sigma_S^2}{N}\mathbf{J}_A - f\frac{\sigma_S^2}{NA}\mathbf{J}_A - f\frac{\sigma_S^2}{N}k^*\mathbf{P}_A + f(1-f)\sigma_A^2 k^*\mathbf{P}_A\right)$$

$$= \frac{N}{n}(1-f)\left(\left[f^2\frac{\sigma_S^2}{N} - f\frac{\sigma_S^2}{NA}\right]\mathbf{J}_A + \left[f(1-f)\sigma_A^2 - f\frac{\sigma_S^2}{N}\right]k^*\mathbf{P}_A\right)$$

$$\mathbf{e}_j'\left(\mathbf{G}\mathbf{V}_{I,II}\right)\mathbf{e}_j = \mathbf{e}_j'\left(\frac{N}{n}(1-f)\left(\left[f^2\frac{\sigma_S^2}{N} - f\frac{\sigma_S^2}{NA}\right]\mathbf{J}_A + \left[f(1-f)\sigma_A^2 - f\frac{\sigma_S^2}{N}\right]k^*\mathbf{P}_A\right)\right)\mathbf{e}_j$$

$$= \frac{N}{n}(1-f)\left(\left[f^2\frac{\sigma_S^2}{N} - f\frac{\sigma_S^2}{NA}\right] + \left[f(1-f)\sigma_A^2 - f\frac{\sigma_S^2}{N}\right]k^*\left(1-\frac{1}{A}\right)\right)$$

$$\mathbf{V}_{II} = f\frac{\sigma_S^2}{N}\left[\mathbf{I}_A - f\mathbf{J}_A\right] + (1-f)^2\sigma_A^2\mathbf{P}_A$$

$$\mathbf{e}_j'\left(\mathbf{V}_{II}\right)\mathbf{e}_j = \mathbf{e}_j'\left(f\frac{\sigma_S^2}{N}\left[\mathbf{I}_A - f\mathbf{J}_A\right] + (1-f)^2\sigma_A^2\mathbf{P}_A\right)\mathbf{e}_j$$

$$= f\frac{\sigma_S^2}{N}\left[1-f\right] + (1-f)^2\sigma_A^2\left(1-\frac{1}{A}\right)$$

Hence,

$$\begin{aligned}
\operatorname{var}\left(\hat{P}_j - P_j\right) &= \mathbf{e}'_j\left(\mathbf{G}\mathbf{V}_I\mathbf{G}' - 2\mathbf{G}\mathbf{V}_{I,II} + \mathbf{V}_{II}\right)\mathbf{e}_j \\
&= N^2\left(\frac{1}{n}-\frac{1}{N}\right)^2\left[\left(\frac{f}{NA}\sigma_S^2 - \frac{f^2}{N}\sigma_S^2\right)+\left(\frac{f}{N}\sigma_S^2 k^* + f^2\sigma_A^2 k^*\right)k^*\left(1-\frac{1}{A}\right)\right] \\
&\quad -2\frac{N}{n}(1-f)\left(\left[f^2\frac{\sigma_S^2}{N} - f\frac{\sigma_S^2}{NA}\right]+\left[f(1-f)\sigma_A^2 - f\frac{\sigma_S^2}{N}\right]k^*\left(1-\frac{1}{A}\right)\right) \\
&\quad + f\frac{\sigma_S^2}{N}(1-f) + (1-f)^2\sigma_A^2\left(1-\frac{1}{A}\right) \\
&= \frac{An(N-n)\sigma_S^2\sigma_A^2 + (N-An)\sigma_S^4}{AnN\left(n\sigma_A^2+\sigma_S^2\right)} \\
&= \frac{An(N-n)\sigma_A^2 + (N-An)\sigma_S^2}{AN\left(n\sigma_A^2+\sigma_S^2\right)}\frac{\sigma_S^2}{n}
\end{aligned}$$

(2) MSE for ANOVA model:

If we use the sample mean of the j^{th} treatment $\left(\bar{Y}_{jI}\right)$ as a predictor for the population mean of the j^{th} treatment P_j, then $MSE\left(\bar{Y}_{jI}\right) = \operatorname{var}\left(\bar{Y}_{jI} - P_j\right)$.

Refer to equation

$$\begin{aligned}
\bar{Y}_{jI} - P_j &= \frac{1}{n}\mathbf{e}'_j \bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^* - \mathbf{e}'_j\frac{1}{N}\left(\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) + \bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right)\mathbf{Y}^* \\
&= \mathbf{e}'_j\left[\frac{1}{n}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right]\mathbf{Y}^* \\
&= \mathbf{e}'_j\left[\left(\frac{1}{n}-\frac{1}{N}\right)\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right]\mathbf{Y}^*
\end{aligned}$$

Define $\mathbf{C}^* = \left(\frac{1}{n}-\frac{1}{N}\right)\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}'_j \otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right) - \frac{1}{N}\bigoplus_{j=1}^{A}\left(\left(\mathbf{1}'_A - \boldsymbol{\delta}'_j\right)\otimes \mathbf{1}'_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)$, plug in $\mathbf{K}_I$ and $\mathbf{K}_{II}$, then

$$\begin{aligned}
\mathbf{C}^* &= \left(\frac{1}{n}-\frac{1}{N}\right)N\mathbf{K}_I - \frac{1}{N}N\mathbf{K}_{II} \\
&= \left(\frac{N}{n}-1\right)\mathbf{K}_I - \mathbf{K}_{II}
\end{aligned}$$

$$\begin{aligned}
\bar{Y}_{jI} - P_j &= \mathbf{e}'_j\mathbf{C}^*\mathbf{Y}^* \\
&= \mathbf{e}'_j\left[\left(\frac{N}{n}-1\right)\mathbf{K}_I - \mathbf{K}_{II}\right]\mathbf{Y}^*
\end{aligned}$$

$$
\begin{aligned}
\operatorname{var}\left(\bar{Y}_{jI}-P_j\right) &= \mathbf{e}_j'\mathbf{C}^* \operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{C}^{*\prime}\mathbf{e}_j \\
&= \mathbf{e}_j'\left[\left(\frac{N}{n}-1\right)\mathbf{K}_I-\mathbf{K}_{II}\right]\operatorname{var}\left(\mathbf{Y}^*\right)\left[\left(\frac{N}{n}-1\right)\mathbf{K}_I'-\mathbf{K}_{II}'\right]\mathbf{e}_j \\
&= \mathbf{e}_j'\left[\left(\frac{N}{n}-1\right)^2\mathbf{K}_I\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_I'-\left(\frac{N}{n}-1\right)\mathbf{K}_{II}\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_I'-\left(\frac{N}{n}-1\right)\mathbf{K}_I\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}'+\mathbf{K}_{II}\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}'\right]\mathbf{e}_j \\
&= \mathbf{e}_j'\left(\left(\frac{N}{n}-1\right)^2\left[\frac{f}{N}\sigma_S^2\left(\mathbf{I}_A-f\mathbf{J}_A\right)+f^2\sigma_A^2\mathbf{P}_A\right]-2\left(\frac{N}{n}-1\right)\left[f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A-\mathbf{I}_A\right]+f(1-f)\sigma_A^2\mathbf{P}_A\right]+\left[f\frac{\sigma_S^2}{N}\left[\mathbf{I}_A-f\mathbf{J}_A\right]+(1-f)^2\sigma_A^2\mathbf{P}_A\right]\right)\mathbf{e}_j \\
&= \left(\frac{N}{n}-1\right)^2\left[\frac{f}{N}\sigma_S^2(1-f)+f^2\sigma_A^2\left(1-\frac{1}{A}\right)\right]-2\left(\frac{N}{n}-1\right)\left[f\frac{\sigma_S^2}{N}(f-1)+f(1-f)\sigma_A^2\left(1-\frac{1}{A}\right)\right]+\left[f\frac{\sigma_S^2}{N}(1-f)+(1-f)^2\sigma_A^2\left(1-\frac{1}{A}\right)\right] \\
&= \left(\frac{N}{n}-1\right)^2\frac{f}{N}\sigma_S^2(1-f)+\left(\frac{N}{n}-1\right)^2 f^2\sigma_A^2\left(1-\frac{1}{A}\right)-2\left(\frac{N}{n}-1\right)f\frac{\sigma_S^2}{N}(f-1)-2\left(\frac{N}{n}-1\right)f(1-f)\sigma_A^2\left(1-\frac{1}{A}\right)+f\frac{\sigma_S^2}{N}(1-f)+(1-f)^2\sigma_A^2\left(1-\frac{1}{A}\right) \\
&= \left(\frac{N}{n}-1\right)^2\frac{f}{N}\sigma_S^2(1-f)-2\left(\frac{N}{n}-1\right)f\frac{\sigma_S^2}{N}(f-1)+f\frac{\sigma_S^2}{N}(1-f)+\left(\frac{N}{n}-1\right)^2 f^2\sigma_A^2\left(1-\frac{1}{A}\right)-2\left(\frac{N}{n}-1\right)f(1-f)\sigma_A^2\left(1-\frac{1}{A}\right)+(1-f)^2\sigma_A^2\left(1-\frac{1}{A}\right) \\
&= \left(\frac{1}{n}-\frac{1}{N}\right)(1-f)^2\sigma_S^2+2(1-f)^2\frac{\sigma_S^2}{N}+\frac{1}{N}f(1-f)\sigma_S^2+(1-f)^2\left(1-\frac{1}{A}\right)\sigma_A^2-2(1-f)^2\left(1-\frac{1}{A}\right)\sigma_A^2+(1-f)^2\sigma_A^2\left(1-\frac{1}{A}\right) \\
&= \left[\left(\frac{1}{n}-\frac{1}{N}\right)(1-f)+2(1-f)\frac{1}{N}+\frac{1}{N}f\right](1-f)\sigma_S^2+\left[(1-f)^2-2(1-f)^2+(1-f)^2\right]\sigma_A^2\left(1-\frac{1}{A}\right) \\
&= \left[\frac{1}{n}-\frac{1}{N}-\frac{1}{N}+f\frac{1}{N}+\frac{2}{N}-\frac{2f}{N}+\frac{1}{N}f\right](1-f)\sigma_S^2 \\
&= \left(\frac{1}{n}-\frac{1}{N}\right)\sigma_S^2
\end{aligned}
$$

Another way to get the MSE of ANOVA:

In order to calculate $MSE\left(\bar{Y}_s\right)$, we need to define another model that we do not permute the treatment. Then the model turns out to be $\mathbf{Y}=\mathbf{U}\mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{U}\boldsymbol{\beta}\mathbf{1}_A'+\mathbf{U}\mathbf{1}_N\boldsymbol{\tau}'+\mathrm{E}$. Then,
$\mathbf{Y}=\mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{B}\mathbf{1}_A'+\mathbf{1}_N\boldsymbol{\tau}'+\mathrm{E}$

$$
\begin{aligned}
E_U\left(\mathbf{Y}\right) &= E_U\left(\mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{B}\mathbf{1}_A'+\mathbf{1}_N\boldsymbol{\tau}'+\mathrm{E}\right) \\
&= \mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{1}_N\boldsymbol{\tau}'+\frac{1}{N}\mathbf{J}_N\boldsymbol{\beta}\mathbf{1}_A' \\
&= \mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{1}_N\boldsymbol{\tau}'
\end{aligned}
$$

$$
\begin{aligned}
vec\left(\mathbf{Y}\right) &= vec\left(\mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{U}\boldsymbol{\beta}\mathbf{1}_A'+\mathbf{1}_N\boldsymbol{\tau}'+\mathbf{U}\boldsymbol{\varepsilon}\right) \\
&= vec\left(\mathbf{1}_N\mathbf{1}_A'\mu+\mathbf{1}_N\boldsymbol{\tau}'+\mathbf{U}\left[\boldsymbol{\beta}\mathbf{1}_A'+\boldsymbol{\varepsilon}\right]\right)
\end{aligned}
$$

$$\begin{aligned} var\left[vec\left(\mathbf{Y}\right)\right] &= \text{var}\left[vec\left(\mathbf{1}_N\mathbf{1}_A'\mu + \mathbf{1}_N\boldsymbol{\tau}' + \mathbf{U}\left[\boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon}\right]\right)\right] \\ &= \text{var}\left[vec\left(\mathbf{U}\left[\boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon}\right]\right)\right] \\ &= \left[\left(\boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon}\right)' \otimes \mathbf{I}_N\right] Var_U\left[vec\left(\mathbf{U}\right)\right]\left[\left(\boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon}\right)' \otimes \mathbf{I}_N\right]' \\ &= \frac{1}{N-1}\left[\left(\mathbf{1}_A\boldsymbol{\beta}' + \boldsymbol{\varepsilon}'\right)\otimes \mathbf{I}_N\right]\left(\mathbf{P}_N \otimes \mathbf{P}_N\right)\left[\left(\boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon}\right)\otimes \mathbf{I}_N\right] \\ &= \frac{1}{N-1}\left[\left(\mathbf{1}_A\boldsymbol{\beta}' + \boldsymbol{\varepsilon}'\right)\mathbf{P}_N\left(\boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon}\right)\right]\otimes \mathbf{P}_N \end{aligned}$$

Notice that

$$\begin{aligned} \mathbf{P}_N\mathbf{y} &= \mathbf{P}_N\mathbf{1}_N\mathbf{1}_A'\mu + \mathbf{P}_N\boldsymbol{\beta}\mathbf{1}_A' + \mathbf{P}_N\mathbf{1}_N\boldsymbol{\tau}' + \mathbf{P}_N\boldsymbol{\varepsilon} \\ &= \mathbf{P}_N\boldsymbol{\beta}\mathbf{1}_A' + \mathbf{P}_N\boldsymbol{\varepsilon} \\ &= \boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon} \end{aligned} .$$

Also, since $\mathbf{P}_N$ is idempotent, then $\mathbf{y}'\mathbf{P}_N\mathbf{y} = \left(\mathbf{P}_N\mathbf{y}\right)'\mathbf{P}_N\left(\mathbf{P}_N\mathbf{y}\right)$. Then, since $\boldsymbol{\Sigma} = \frac{1}{N-1}\mathbf{y}'\mathbf{P}_N\mathbf{y}$,

$\boldsymbol{\Sigma} = \frac{1}{N-1}\left(\mathbf{1}_A\boldsymbol{\beta}' + \boldsymbol{\varepsilon}'\right)\mathbf{P}_N\left(\boldsymbol{\beta}\mathbf{1}_A' + \boldsymbol{\varepsilon}\right)$. As a result,

$$Var_{U|V}\left[vec\left(\mathbf{Y}\right)\right] = \boldsymbol{\Sigma}\otimes\mathbf{P}_N$$

In order to simplify the problem, we first assume $\varepsilon_{sa} = 0$ (No interaction between effect of units and effect of treatments. This implies the treatment effect will not change with respect to different unit and the unit effect will not change with respect to different treatments). With this assumption we simplify the model, $\boldsymbol{\Sigma}$ can be simplified as $\boldsymbol{\Sigma} = \sigma_S^2\mathbf{J}_A$. Substitute $\boldsymbol{\Sigma} = \sigma_S^2\mathbf{J}_A$ to equation, then

$$Var_{UV}\left[vec\left(\mathbf{Y}\right)\right] = \sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N$$

MSE for ANOVA model:
If we use the sample mean of the s treatment $\left(\bar{Y}_{sI}\right)$ as a estimator for the population mean of the s treatment $\bar{Y}_s$, then $MSE\left(\bar{Y}_{sI}\right) = \text{var}\left(\bar{Y}_{sI} - \bar{Y}_s\right)$.

Refer to equation

$$\begin{aligned}\bar{Y}_{sI}-\bar{Y}_s &= \frac{1}{n}\mathbf{e}_s'\bigoplus_{s=1}^{A}\left(\boldsymbol{\delta}_s'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\mathbf{Y}^* - \mathbf{e}_s'\frac{1}{N}\left(\bigoplus_{s=1}^{A}\left(\boldsymbol{\delta}_s'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)+\bigoplus_{s=1}^{A}\left(\left(\mathbf{1}_A'-\boldsymbol{\delta}_s'\right)\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right)\mathbf{Y}^*\\
&= \mathbf{e}_s'\left[\frac{1}{n}\bigoplus_{s=1}^{A}\left(\boldsymbol{\delta}_s'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)-\frac{1}{N}\bigoplus_{s=1}^{A}\left(\boldsymbol{\delta}_s'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)-\frac{1}{N}\bigoplus_{s=1}^{A}\left(\left(\mathbf{1}_A'-\boldsymbol{\delta}_s'\right)\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right]\mathbf{Y}^*\\
&= \mathbf{e}_s'\left[\left(\frac{1}{n}-\frac{1}{N}\right)\bigoplus_{s=1}^{A}\left(\boldsymbol{\delta}_s'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)-\frac{1}{N}\bigoplus_{s=1}^{A}\left(\left(\mathbf{1}_A'-\boldsymbol{\delta}_s'\right)\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)\right]\mathbf{Y}^*\end{aligned}$$

Define $\mathbf{C}^* = \left(\frac{1}{n}-\frac{1}{N}\right)\bigoplus_{s=1}^{A}\left(\boldsymbol{\delta}_s'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)-\frac{1}{N}\bigoplus_{s=1}^{A}\left(\left(\mathbf{1}_A'-\boldsymbol{\delta}_s'\right)\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{1}'}\right)$, plug in $\mathbf{K}_I$ and $\mathbf{K}_{II}$, then

$$\begin{aligned}\mathbf{C}^* &= \left(\frac{1}{n}-\frac{1}{N}\right)N\mathbf{K}_I-\frac{1}{N}N\mathbf{K}_{II}\\
&= \left(\frac{N}{n}-1\right)\mathbf{K}_I-\mathbf{K}_{II}\end{aligned}$$

$$\begin{aligned}\bar{Y}_{sI}-\bar{Y}_s &= \mathbf{e}_s'\mathbf{C}^*\mathbf{Y}^*\\
&= \mathbf{e}_s'\left[\left(\frac{N}{n}-1\right)\mathbf{K}_I-\mathbf{K}_{II}\right]\mathbf{Y}^*\end{aligned}$$

$$\begin{aligned}\operatorname{var}\left(\bar{Y}_{sI}-\bar{Y}_s\right) &= \mathbf{e}_s'\mathbf{C}^*\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{C}^{*\prime}\mathbf{e}_s\\
&= \mathbf{e}_s'\left[\left(\frac{N}{n}-1\right)\mathbf{K}_I-\mathbf{K}_{II}\right]\operatorname{var}\left(\mathbf{Y}^*\right)\left[\left(\frac{N}{n}-1\right)\mathbf{K}_I'-\mathbf{K}_{II}'\right]\mathbf{e}_s\\
&= \mathbf{e}_s'\left[\left(\frac{N}{n}-1\right)^2\mathbf{K}_I\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_I'-2\left(\frac{N}{n}-1\right)\mathbf{K}_{II}\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_I'+\mathbf{K}_{II}\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}'\right]\mathbf{e}_s\end{aligned}$$

$$\begin{aligned}\mathbf{K}_I\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_I' &= \frac{1}{N}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}}\right)\operatorname{var}\left(\mathbf{Y}^*\right)\frac{1}{N}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j\otimes\mathbf{1}_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\\
&= \frac{1}{N}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}}\right)\sigma_S^2\mathbf{J}_A\otimes\mathbf{P}_N\frac{1}{N}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j\otimes\mathbf{1}_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\\
&= \frac{\sigma_S^2}{N^2}\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j'\otimes\mathbf{1}_n' \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}}\right)\left(\mathbf{J}_A\otimes\mathbf{P}_N\right)\bigoplus_{j=1}^{A}\left(\boldsymbol{\delta}_j\otimes\mathbf{1}_n \;\vdots\; \underset{1\times(N-nA)}{\mathbf{0}'}\right)\\
&= \frac{f}{N}\sigma_S^2\left(\mathbf{I}_A-f\mathbf{J}_A\right)\end{aligned}$$

$$\mathbf{K}_{II}\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_I' == f\,\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A-\mathbf{I}_A\right]$$

$$\mathbf{K}_{II}\operatorname{var}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}' = f\,\frac{\sigma_S^2}{N}\left[\mathbf{I}_A-f\mathbf{J}_A\right]$$

Hence,

$$\begin{aligned}
\text{var}\left(\bar{Y}_{sI}-\bar{Y}_{s}\right) &= \mathbf{e}_s'\left[\left(\frac{N}{n}-1\right)^2 \mathbf{K}_I \text{ var}\left(\mathbf{Y}^*\right)\mathbf{K}_I' - 2\left(\frac{N}{n}-1\right)\mathbf{K}_{II} \text{ var}\left(\mathbf{Y}^*\right)\mathbf{K}_I' + \mathbf{K}_{II} \text{ var}\left(\mathbf{Y}^*\right)\mathbf{K}_{II}'\right]\mathbf{e}_s \\
&= \mathbf{e}_s'\left[\left(\frac{N}{n}-1\right)^2 \frac{f}{N}\sigma_S^2\left(\mathbf{I}_A - f\mathbf{J}_A\right) - 2\left(\frac{N}{n}-1\right) f\,\frac{\sigma_S^2}{N}\left(f\mathbf{J}_A - \mathbf{I}_A\right) + f\,\frac{\sigma_S^2}{N}\left(\mathbf{I}_A - f\mathbf{J}_A\right)\right]\mathbf{e}_s \\
&= \mathbf{e}_s'\sigma_S^2\left[\left(\frac{N}{n}-1\right)^2 \frac{f}{N}\left(\mathbf{I}_A - f\mathbf{J}_A\right) - 2\left(\frac{N}{n}-1\right) f\,\frac{1}{N}\left(f\mathbf{J}_A - \mathbf{I}_A\right) + f\,\frac{1}{N}\left(\mathbf{I}_A - f\mathbf{J}_A\right)\right]\mathbf{e}_s \\
&= \sigma_S^2\left[\left(\frac{N}{n}-1\right)^2 \frac{f}{N}(1-f) - 2\left(\frac{N}{n}-1\right) f\,\frac{1}{N}(f-1) + f\,\frac{1}{N}(1-f)\right] \\
&= \sigma_S^2(1-f)\, f\,\frac{1}{N}\left[\left(\frac{N}{n}-1\right)^2 + 2\left(\frac{N}{n}-1\right)+1\right] \\
&= \sigma_S^2(1-f)\, f\,\frac{1}{N}\left(\frac{N}{n}-1+1\right)^2 \\
&= \sigma_S^2(1-f)\frac{1}{N}\left(\frac{N}{n}\right) \\
&= \sigma_S^2\left(\frac{1}{n}-\frac{1}{N}\right)
\end{aligned}$$

This is the same as the result for $MSE\left(\bar{Y}_{jI}\right)$.

Comparing $MSE\left(\bar{Y}_{jI}\right)$ and $MSE\left(\bar{Y}_{sI}\right)$

Compare $\text{var}\left(\hat{P}_j - P_j\right) = \dfrac{An(N-n)\sigma_S^2\sigma_A^2 + (N-An)\sigma_S^4}{An\left(n\sigma_A^2+\sigma_S^2\right)}$ and $\text{var}\left(\bar{Y}_{jI} - P_j\right) = \left(\dfrac{1}{n}-\dfrac{1}{N}\right)\sigma_S^2 = \dfrac{N-n}{Nn}\sigma_S^2$

$$\begin{aligned}
\text{var}\left(\hat{P}_j - P_j\right)/\text{var}\left(\bar{Y}_{jI} - P_j\right) &= \frac{An(N-n)\sigma_S^2\sigma_A^2 + (N-An)\sigma_S^4}{AnN\left(n\sigma_A^2+\sigma_S^2\right)}\frac{Nn}{(N-n)\sigma_S^2} \\
&= \frac{An\sigma_A^2 + \left(\dfrac{N-An}{N-n}\right)\sigma_S^2}{An\sigma_A^2 + A\sigma_S^2} < 1
\end{aligned}$$

Simplification of $\hat{P}_j$

$\hat{P}_j = f\bar{Y}_{jI} + (1-f)\left(\bar{\bar{Y}}_I + k^*\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right)\right)$ where $k^* = \dfrac{f}{1-f}\dfrac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}$

$$
\begin{aligned}
\hat{P}_j &= f\bar{Y}_{jI} + (1-f)\left(\bar{\bar{Y}}_I + k^*\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right)\right) \\
&= f\bar{Y}_{jI} + (1-f)\left(\bar{\bar{Y}}_I + \frac{f}{1-f}\frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right)\right) \\
&= f\bar{Y}_{jI} + (1-f)\bar{\bar{Y}}_I + f\,\frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right) \\
&= f\left(1 + \frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}\right)\bar{Y}_{jI} + \left(1 - f\left(1 + \frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}\right)\right)\bar{\bar{Y}}_I \\
&= \frac{n}{N}\frac{N\sigma_A^2}{\sigma_S^2 + n\sigma_A^2}\bar{Y}_{jI} + \left(1 - \frac{n}{N}\frac{N\sigma_A^2}{\sigma_S^2 + n\sigma_A^2}\right)\bar{\bar{Y}}_I \\
&= \frac{n\sigma_A^2}{\sigma_S^2 + n\sigma_A^2}\bar{Y}_{jI} + \left(1 - \frac{n\sigma_A^2}{\sigma_S^2 + n\sigma_A^2}\right)\bar{\bar{Y}}_I \\
&= \bar{\bar{Y}}_I + \frac{n\sigma_A^2}{\sigma_S^2 + n\sigma_A^2}\left(\bar{Y}_{jI} - \bar{\bar{Y}}_I\right)
\end{aligned}
$$

Evaluate the Contrast of $P_j - P_{j*}$:

Target: $P_j - P_{j*} = \frac{1}{N}\sum_{i=1}^{N} Y_{ij} - \frac{1}{N}\sum_{i=1}^{N} Y_{ij*}$

Collapsing: $\begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \begin{pmatrix} f\bar{\mathbf{Y}}_I \\ (1-f)\bar{\mathbf{Y}}_{II} \end{pmatrix}$

We know $E_{UV}\begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \left[\begin{pmatrix} f \\ 1-f \end{pmatrix} \otimes \mathbf{1}_A\right]\mu$ and $\begin{pmatrix} \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}'_{I,II} & \mathbf{V}_{II} \end{pmatrix}$

Now $\begin{pmatrix} \mathbf{g}'_I & \mathbf{g}'_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_j - \mathbf{e}'_{j*} & \mathbf{e}'_j - \mathbf{e}'_{j*} \end{pmatrix}$, $\mathbf{X}_I = f\mathbf{1}_A$ (17) and $\mathbf{X}_{II} = (1-f)\mathbf{1}_A$ (18).

To evaluate the predictor, we find note $\mathbf{V}_I^{-1}$ first.

Since we know:

$$
\begin{aligned}
\mathbf{V}_I &= \frac{f}{N}\sigma_S^2\left(\mathbf{I}_A - f\mathbf{J}_A\right) + f^2\sigma_A^2\mathbf{P}_A \\
&= \frac{f}{N}\sigma_S^2\left(\mathbf{I}_A - f\mathbf{J}_A\right) + f^2\sigma_A^2\left(\mathbf{I}_A - \frac{1}{A}\mathbf{J}_A\right) \\
&= \left(\frac{f}{N}\sigma_S^2 + f^2\sigma_A^2\right)\mathbf{I}_A - \left(\frac{f^2\sigma_A^2}{A} + \frac{f^2\sigma_S^2}{N}\right)\mathbf{J}_A
\end{aligned}
$$

Define $a = \left(\frac{f}{N}\sigma_S^2 + f^2\sigma_A^2\right)$ and $b = \left(\frac{f^2\sigma_A^2}{A} + \frac{f^2\sigma_S^2}{N}\right)$, plug in $\mathbf{V}_I$, then

$$\mathbf{V}_I = a\mathbf{I}_A - b\mathbf{J}_A = a\left(\mathbf{I}_A - \frac{b}{a}\mathbf{J}_A\right)$$

Hence,

$$\mathbf{V}_I^{-1} = a^{-1}\left(\mathbf{I}_A + \frac{1}{\frac{a}{b} - A}\mathbf{J}_A\right) = \frac{1}{a}\mathbf{I}_A + \frac{b}{a^2 - Aab}\mathbf{J}_A \qquad (19)$$

Other terms simplify:
Using (12) and (14),

$$\begin{aligned}\hat{\alpha} &= \left(\mathbf{X}_I'\mathbf{V}_I^{-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{-1}\mathbf{Y}_I^* \\ &= \left[f\mathbf{1}_A'\left(\frac{1}{a}\mathbf{I}_A + \frac{b}{a^2 - Aab}\mathbf{J}_A\right)f\mathbf{1}_A\right]^{-1}\left(f\mathbf{1}_A'\right)\left(\frac{1}{a}\mathbf{I}_A + \frac{b}{a^2 - Aab}\mathbf{J}_A\right)f\bar{\mathbf{Y}}_I \\ &= \left(\frac{Af^2}{a - Ab}\right)^{-1}\frac{f^2}{a - Ab}\mathbf{1}_A'\bar{\mathbf{Y}}_I \qquad (20) \\ &= \frac{1}{A}\mathbf{1}_A'\bar{\mathbf{Y}}_I \\ &= \bar{\bar{Y}}_I\end{aligned}$$

where $\bar{\bar{Y}}_I$ is the overall sample mean.

Next we simplify the term $\mathbf{V}_{II,I}\mathbf{V}_I^{-1}$. We know:

$$\begin{aligned}\mathbf{V}_{I,II} &= f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A - \mathbf{I}_A\right] + f(1-f)\sigma_A^2\mathbf{P}_A \\ &= f\frac{\sigma_S^2}{N}\left[f\mathbf{J}_A - \mathbf{I}_A\right] + f(1-f)\sigma_A^2\left(\mathbf{I}_A - \frac{1}{A}\mathbf{J}_A\right) \\ &= \left(f(1-f)\sigma_A^2 - f\frac{\sigma_S^2}{N}\right)\mathbf{I}_A + \left(\frac{\sigma_S^2 f^2}{N} - \frac{f(1-f)\sigma_A^2}{A}\right)\mathbf{J}_A\end{aligned}$$

Define $c = f(1-f)\sigma_A^2 - f\frac{\sigma_S^2}{N}$ and $d = \frac{\sigma_S^2 f^2}{N} - \frac{f(1-f)\sigma_A^2}{A}$

Hence, $\mathbf{V}_{I,II} = c\mathbf{I}_A + d\mathbf{J}_A$

Then

$$
\begin{aligned}
\mathbf{V}_{II,I}\mathbf{V}_I^{-1} = \mathbf{V}'_{I,II}\mathbf{V}_I^{-1} &= \left(c\mathbf{I}_A + d\mathbf{J}_A\right)'\left(\frac{1}{a}\mathbf{I}_A + \frac{b}{a^2 - Aab}\mathbf{J}_A\right) \\
&= \left(\frac{c}{a}\mathbf{I}_A + \frac{bc}{a^2 - Aab}\mathbf{J}_A + \frac{d}{a}\mathbf{J}_A + \frac{bdA}{a^2 - Aab}\mathbf{J}_A\right) \\
&= \left[\frac{c}{a}\mathbf{I}_A + \left(\frac{bc + d(a - Ab) + bdA}{a^2 - Aab}\right)\mathbf{J}_A\right] \\
&= \left[\frac{c}{a}\mathbf{I}_A + \left(\frac{bc + ad}{a^2 - Aab}\right)\mathbf{J}_A\right]
\end{aligned}
\qquad (21).
$$

Using (13), (15) and (16),

$$
\begin{aligned}
\hat{P}_j - \hat{P}_{j*} &= \mathbf{g}_I{}'\mathbf{Y}_I^* + \mathbf{g}_{II}{}'\left[\mathbf{X}_{II}\hat{\alpha} + \mathbf{V}_{II,I}\mathbf{V}_I^{-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] \\
&= \left(\mathbf{e}'_j - \mathbf{e}'_{j*}\right)\mathbf{Y}_I^* + \left(\mathbf{e}'_j - \mathbf{e}'_{j*}\right)\left[(1-f)\bar{\bar{Y}}_I\mathbf{1}_A + \left[\frac{c}{a}\mathbf{I}_A + \left(\frac{bc+ad}{a^2 - Aab}\right)\mathbf{J}_A\right]\left(\mathbf{Y}_I^* - f\bar{\bar{Y}}_I\mathbf{1}_A\right)\right] \\
&= f\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) + \mathbf{e}'_j\left[(1-f)\bar{\bar{Y}}_I\mathbf{1}_A + \left[\frac{c}{a}\mathbf{I}_A + \left(\frac{bc+ad}{a^2 - Aab}\right)\mathbf{J}_A\right]\left(f\bar{\mathbf{Y}}_I - f\bar{\bar{Y}}_I\mathbf{1}_A\right)\right] \\
&\quad -\mathbf{e}'_{j*}\left[(1-f)\bar{\bar{Y}}_I\mathbf{1}_A + \left[\frac{c}{a}\mathbf{I}_A + \left(\frac{bc+ad}{a^2 - Aab}\right)\mathbf{J}_A\right]\left(\mathbf{Y}_I^* - f\bar{\bar{Y}}_I\mathbf{1}_A\right)\right] \\
&= f\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) + \left((1-f)\bar{\bar{Y}}_I + \left[\frac{c}{a}\mathbf{e}'_j + \left(\frac{bc+ad}{a^2 - Aab}\right)\mathbf{1}'_A\right]\left(f\bar{\mathbf{Y}}_I - f\bar{\bar{Y}}_I\mathbf{1}_A\right)\right) \\
&\quad -\left((1-f)\bar{\bar{Y}}_I + \left[\frac{c}{a}\mathbf{e}'_{j*} + \left(\frac{bc+ad}{a^2 - Aab}\right)\mathbf{1}'_A\right]\left(f\bar{\mathbf{Y}}_I - f\bar{\bar{Y}}_I\mathbf{1}_A\right)\right) \\
&= f\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) + \frac{c}{a}\left(\mathbf{e}'_j - \mathbf{e}'_{j*}\right)\left(f\bar{\mathbf{Y}}_I - f\bar{\bar{Y}}_I\mathbf{1}_A\right) \\
&= f\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) + \frac{c}{a}f\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) \\
&= f\left(1 + \frac{c}{a}\right)\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right)
\end{aligned}
$$

Since $\frac{c}{a} = \frac{(N-n)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}$, hence

$$\begin{aligned}
\hat{P}_j - \hat{P}_{j*} &= f\left(1+\frac{c}{a}\right)\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) \\
&= f\left(1+\frac{\left(N-n\right)\sigma_A^2 - \sigma_S^2}{\sigma_S^2 + n\sigma_A^2}\right)\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) \\
&= \frac{n}{N}\left(\frac{N\sigma_A^2}{\sigma_S^2 + n\sigma_A^2}\right)\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right) \\
&= \frac{n\sigma_A^2}{\sigma_S^2 + n\sigma_A^2}\left(\bar{Y}_{jI} - \bar{Y}_{j*I}\right)
\end{aligned}$$