Simple Random Sampling with Measurement Error
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1. FINITE POPULATION

Frames

Labels

Expected Response

Response

Notation and terminology

Expected Response: y_s $s = 1,...,N$

Response: $Y_{sk} = y_s + W_{sk}$ $k = 1,...,r_s$ index of order of measure

Assumptions:

R Subscript representing expectation with respect to replication.

$E_R\left(W_{sk}\right) = 0$ for all $s = 1,...,N$, $k = 1,...,r_s$.

$\text{var}_R\left(W_{sk}\right) = \sigma_{se}^2$ for all $s = 1,...,N$, $k = 1,...,r_s$.

$\text{var}_R\left(W_{sk}, W_{s*k*}\right) = 0$ for all $s = 1,...,N$; $s* = 1,...,N$; $k = 1,...,r_s$; $k*\left(\neq k\right) = 1,...,r_s$.

Parameters:

$$\mu = \frac{1}{N}\sum_{s=1}^{N} y_s,\ \beta_s = y_s - \mu,\quad \sigma^2 = \frac{1}{N-1}\sum_{s=1}^{N}\left(y_s - \mu\right)^2,\ \sigma_e^2 = \frac{1}{N}\sum_{s=1}^{N}\sigma_{se}^2 .$$

Additional notation:

Define $\mathbf{U}_i = \begin{pmatrix} U_{i1} & U_{i2} & \cdots & U_{iN} \end{pmatrix}'$,

$\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_N \end{pmatrix}'$,

$\mathbf{Y}_R = \left(\left(Y_{sk}\right)\right) = \begin{pmatrix} Y_{1k} & Y_{2k} & \cdots & Y_{Nk} \end{pmatrix}'$ $\mathbf{W} = \left(\left(W_{sk}\right)\right) = \begin{pmatrix} W_{1k} & W_{2k} & \cdots & W_{Nk} \end{pmatrix}'$ and

$\mathbf{U}' = \left(\left(\mathbf{U}_i\right)\right) = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 & \cdots & \mathbf{U}_N \end{pmatrix}$.

2. SIMPLIFICATIONS

Assume $r_s = 1$ for all $s = 1,...,N$ so that $k = 1$. (1 measure of response)

Note: If $k = 1,...,r_s$, we could define $\bar{Y}_{s1} = \frac{1}{r_s}\sum_{k=1}^{r_s} Y_{sk}$ and $\text{var}\left(\bar{Y}_{s1}\right) = \frac{\sigma_{se}^2}{r_s}$ so that $\bar{Y}_{s1} = y_s + \bar{E}_{s1}$ where $\bar{E}_{s1} = \frac{1}{r_s}\sum_{k=1}^{r_s} E_{sk}$, $E\left(\bar{E}_{s1}\right) = 0$ and $\text{var}\left(\bar{E}_{s1}\right) = \frac{\sigma_{se}^2}{r_s}$. Thus, with suitable definitions, the model we discuss is more general.

3. PERMUTATION

Assume the subjects are permuted. There are two possibilities for representing the permutations plus error.

$Y_{ik}^{*} = \mathbf{U}_i'\left(\mathbf{y}+\mathbf{W}\right)$ represents <u>measurement error</u> on a subject

$\tilde{Y}_{ik} = \mathbf{U}_i'\mathbf{y}+\tilde{W}_{ik}$ represents <u>interviewer error</u> associated with position i

$W_{ik}^{*} = \sum_{s=1}^{N} U_{is}W_{sk}$ or $W_{ik}^{*} = \mathbf{U}_i'\mathbf{W}$

<u>We only discuss measurement error</u>.

Also, define

$$\mathbf{Y} = \left(\left(Y_i\right)\right) = \mathbf{U}\mathbf{y}$$

$$\mathbf{W}^{*} = \left(\left(W_{ik}^{*}\right)\right) = \mathbf{U}\mathbf{W}$$

$$\mathbf{Y}^{*} = \left(\left(Y_{ik}^{*}\right)\right) = \mathbf{Y}+\mathbf{W}^{*}$$

$$\mathbf{Y}^{*} = vec\left(\left(\mathbf{y}+\mathbf{W}\right)'\mathbf{U}'\right) = \left(\mathbf{I}_N \otimes \left(\mathbf{y}+\mathbf{W}\right)'\right)vec\left(\mathbf{U}'\right)$$

4. DEFINE THE EXPANDED MODEL

Let $\overleftrightarrow{Y}_{isk} = U_{is}Y_{sk} = U_{is}\left(y_s + W_{sk}\right) = \overleftrightarrow{Y}_{is} + \overleftrightarrow{W}_{isk}$.

Define

$$\overleftrightarrow{\mathbf{Y}}_i = \left(\bigoplus_{s=1}^{N} y_s\right)\mathbf{U}_i$$

$$\overleftrightarrow{\mathbf{W}}_i = \left(\bigoplus_{s=1}^{N} W_{sk}\right)\mathbf{U}_i$$

$$\overleftrightarrow{\mathbf{Y}}_i^{*} = \overleftrightarrow{\mathbf{Y}}_i + \overleftrightarrow{\mathbf{W}}_i = \left(\bigoplus_{s=1}^{N}\left(y_s + W_{sk}\right)\right)\mathbf{U}_i = \begin{pmatrix} U_{i1}\left(y_1+W_{1k}\right) & U_{i2}\left(y_2+W_{2k}\right) & \cdots & U_{iN}\left(y_N+W_{Nk}\right)\end{pmatrix}'$$

Define the expanded random variables as

$$\overleftrightarrow{\mathbf{Y}}^{*} = vec\begin{pmatrix} U_{11}Y_{1k} & U_{21}Y_{1k} & \cdots & U_{N1}Y_{1k} \\ U_{12}Y_{2k} & U_{22}Y_{2k} & \cdots & U_{N2}Y_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ U_{1N}Y_{Nk} & U_{2N}Y_{Nk} & \cdots & U_{NN}Y_{Nk} \end{pmatrix}$$

$$\overleftrightarrow{\mathbf{Y}}^{*} = vec\begin{pmatrix}\overleftrightarrow{\mathbf{Y}}_1^{*} & \overleftrightarrow{\mathbf{Y}}_2^{*} & \cdots & \overleftrightarrow{\mathbf{Y}}_N^{*}\end{pmatrix}$$

$$\overleftrightarrow{\mathbf{Y}}^{*} = \left[\mathbf{I}_N \otimes \left(\bigoplus_{s=1}^{N}\left(y_s + W_{sk}\right)\right)\right]vec\left(\mathbf{U}'\right)$$

5. EXPECTED VALUE AND VARIANCE

Denote expected value with respect to sampling via a subscript ξ, and expected value with respect to measurement error via a subscript R.

<u>Expected Value</u>:

$$E_R\left(\vec{\mathbf{Y}}^*\right)=\left[\mathbf{I}_N\otimes\left(\bigoplus_{s=1}^{N}y_s\right)\right]vec\left(\mathbf{U}'\right)=\vec{\mathbf{Y}}$$

$$E_\xi\left(\vec{\mathbf{Y}}\right)=\left[\frac{1}{N}\left(\mathbf{1}_N\otimes\mathbf{I}_N\right)\right]\mathbf{y}$$

$$E_{\xi R}\left(\vec{\mathbf{Y}}^*\right)=\left[\frac{1}{N}\left(\mathbf{1}_N\otimes\mathbf{I}_N\right)\right]\mathbf{y}$$

<u>Variance</u>:

$$\text{var}_{\xi R}\left(\vec{\mathbf{Y}}^*\right)=E_\xi\left[\text{var}_{R|\xi}\left(\vec{\mathbf{Y}}^*\right)\right]+\text{var}_\xi\left[E_{R|\xi}\left(\vec{\mathbf{Y}}^*\right)\right].$$

Now

$$E_{R|\xi}\left(\vec{\mathbf{Y}}^*\right)=\vec{\mathbf{Y}}\text{ and}$$

$$\text{var}_\xi\left(\vec{\mathbf{Y}}\right)=\mathbf{P}_N\otimes\mathbf{\Delta}\text{ where }\mathbf{\Delta}=\frac{1}{N-1}\left(\bigoplus_{s=1}^{N}y_s\right)\mathbf{P}_N\left(\bigoplus_{s=1}^{N}y_s\right).$$

As a result,

$$\text{var}_\xi\left[E_{R|\xi}\left(\vec{\mathbf{Y}}^*\right)\right]=\mathbf{P}_N\otimes\mathbf{\Delta}.$$

Also,

$$\text{var}_{R|\xi}\left(\vec{\mathbf{Y}}^*\right)=\text{var}_{R|\xi}\left[vec\left(\vec{\mathbf{Y}}_1^*\quad\vec{\mathbf{Y}}_2^*\quad\cdots\quad\vec{\mathbf{Y}}_N^*\right)\right]$$

$$=\begin{pmatrix}\text{var}_{R|\xi}\left(\vec{\mathbf{Y}}_1^*\right) & \text{cov}_{R|\xi}\left(\vec{\mathbf{Y}}_1^*,\vec{\mathbf{Y}}_2'^*\right) & \cdots & \text{cov}_{R|\xi}\left(\vec{\mathbf{Y}}_1^*,\vec{\mathbf{Y}}_N'^*\right)\\ \text{cov}_{R|\xi}\left(\vec{\mathbf{Y}}_2^*,\vec{\mathbf{Y}}_1'^*\right) & \text{var}_{R|\xi}\left(\vec{\mathbf{Y}}_2^*\right) & \cdots & \text{cov}_{R|\xi}\left(\vec{\mathbf{Y}}_2^*,\vec{\mathbf{Y}}_N'^*\right)\\ \vdots & \vdots & \ddots & \vdots\\ \text{cov}_{R|\xi}\left(\vec{\mathbf{Y}}_N^*,\vec{\mathbf{Y}}_1'^*\right) & \text{cov}_{R|\xi}\left(\vec{\mathbf{Y}}_N^*,\vec{\mathbf{Y}}_2'^*\right) & \cdots & \text{var}_{R|\xi}\left(\vec{\mathbf{Y}}_N^*\right)\end{pmatrix}.$$

Now

$$\text{var}_{R|\xi}\left(\vec{\mathbf{Y}}_i^*\right)=\text{var}_{R|\xi}\left[\left(U_{i1}\left(y_1+W_{1k}\right)\quad U_{i2}\left(y_2+W_{2k}\right)\quad\cdots\quad U_{iN}\left(y_N+W_{Nk}\right)\right)'\right]$$

$$=\text{var}_{R|\xi}\left[\left(\bigoplus_{s=1}^{N}U_{is}\right)\begin{pmatrix}W_{1k}\\W_{2k}\\\vdots\\W_{Nk}\end{pmatrix}\right]$$

$$=\bigoplus_{s=1}^{N}U_{is}\sigma_{se}^2$$

while $\text{cov}_{R|\xi}\left(\vec{\mathbf{Y}}_1^*,\vec{\mathbf{Y}}_2'^*\right)=\mathbf{0}_N$. As a result, $\text{var}_{R|\xi}\left(\vec{\mathbf{Y}}^*\right)=\mathbf{I}_N\otimes\left(\bigoplus_{s=1}^{N}U_{is}\sigma_{se}^2\right)$ and

$E_\xi\left[\text{var}_{R|\xi}\left(\vec{\mathbf{Y}}^*\right)\right]=\frac{1}{N}\mathbf{I}_N\otimes\left(\bigoplus_{s=1}^{N}\sigma_{se}^2\right)$. Using this result,

$$\text{var}_{\xi R}\left(\overleftrightarrow{\mathbf{Y}}^*\right)=\frac{1}{N}\mathbf{I}_N\otimes\left(\bigoplus_{s=1}^{N}\sigma_{se}^2\right)+\mathbf{P}_N\otimes\boldsymbol{\Delta}.$$

6. COLLAPSING THE EXPANDED RANDOM VARIABLES

Linear combinations of the expanded random variables correspond to constants and random variables that are commonly used.

a. Collapsing to $\mathbf{Y}_R$: $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\mathbf{Y}_R$ where $\mathbf{L}=\mathbf{1}_N'\otimes\mathbf{I}_N$ (population frame with measurement error)

b. Collapsing to $\mathbf{Y}^*$: $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\mathbf{Y}^*$ where $\mathbf{L}=\mathbf{I}_N\otimes\mathbf{1}_N'$ (usual random variables)

c. Collapsing to sample and remainder totals $\begin{pmatrix}Y_I^*\\Y_{II}^*\end{pmatrix}$: $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\begin{pmatrix}Y_I^*\\Y_{II}^*\end{pmatrix}$ where $\mathbf{L}=\begin{pmatrix}\mathbf{1}_n' & \mathbf{0}_{N-n}'\\ \mathbf{0}_n' & \mathbf{1}_{N-n}'\end{pmatrix}\otimes\mathbf{1}_N'$

where $Y_I^*=\sum_{i=1}^{n}Y_{ik}^*$ and $Y_{II}^*=\sum_{i=n+1}^{N}Y_{ik}^*$.

d. Collapsing to sample and remainder means $\begin{pmatrix}\bar{Y}_I^*\\\bar{Y}_{II}^*\end{pmatrix}$: $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\begin{pmatrix}\bar{Y}_I^*\\\bar{Y}_{II}^*\end{pmatrix}$ where

$$\mathbf{L}=\frac{1}{N}\begin{pmatrix}\frac{1}{f}\mathbf{1}_n' & \mathbf{0}_{N-n}'\\ \mathbf{0}_n' & \frac{1}{1-f}\mathbf{1}_{N-n}'\end{pmatrix}\otimes\mathbf{1}_N' \text{ and } \bar{Y}_I^*=\frac{1}{n}\sum_{i=1}^{n}Y_{ik}^* \text{ and } \bar{Y}_{II}^*=\frac{1}{N-n}\sum_{i=n+1}^{N}Y_{ik}^*.$$

e. Collapsing to weighted population frame: $\mathbf{Y}_{wR}$: $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\mathbf{Y}_{wR}$ where $\mathbf{L}=\mathbf{1}_N'\otimes\left(\bigoplus_{s=1}^{N}w_s\right)$ where $\sum_{s=1}^{N}w_s=1$, $\mathbf{Y}_{wR}=\left(\left(Y_{wsk}\right)\right)$ and $Y_{wsk}=w_sY_{sk}$.

f. Collapsing to weighted population random variables: $\mathbf{Y}_w^*$: $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\mathbf{Y}_w^*$ where $\mathbf{L}=\mathbf{1}_N'\otimes\mathbf{w}'$ where $\mathbf{w}=\left(\left(w_s\right)\right)=\begin{pmatrix}w_1 & w_2 & \cdots & w_N\end{pmatrix}'$, $\sum_{s=1}^{N}w_s=1$, $\mathbf{Y}_w^*=\left(\left(Y_{wik}^*\right)\right)$ and $Y_{wik}^*=\sum_{s=1}^{N}U_{is}Y_{wsk}$.

g. Collapsing to weighted sample and remainder totals $\begin{pmatrix}Y_{wI}^*\\Y_{wII}^*\end{pmatrix}$: $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\begin{pmatrix}Y_{wI}^*\\Y_{wII}^*\end{pmatrix}$ where

$\mathbf{L}=\begin{pmatrix}\mathbf{1}_n' & \mathbf{0}_{N-n}'\\ \mathbf{0}_n' & \mathbf{1}_{N-n}'\end{pmatrix}\otimes\mathbf{w}'$ where $\mathbf{w}=\left(\left(w_s\right)\right)=\begin{pmatrix}w_1 & w_2 & \cdots & w_N\end{pmatrix}'$, $\sum_{s=1}^{N}w_s=1$, $\mathbf{Y}_w^*=\left(\left(Y_{wik}^*\right)\right)$,

$Y_{wI}^*=\sum_{i=1}^{n}Y_{wik}^*$ and $Y_{wII}^*=\sum_{i=n+1}^{N}Y_{wik}^*$.

7. PARTITIONING INTO THE SAMPLE AND REMAINDER

Collapsed random variables that are a linear combination of $\mathbf{Y}_R$, $\mathbf{Y}_{wR}$, $\mathbf{Y}^*$ or $\mathbf{Y}_w^*$ can be partitioned into a sample and a remainder (collapsing a-g in Section 6) by partitioning

$\mathbf{L}=\begin{pmatrix}\mathbf{L}_I\\ \mathbf{L}_{II}\end{pmatrix}$. In general, we represent the result as $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^*=\begin{pmatrix}\mathbf{L}_I\overleftrightarrow{\mathbf{Y}}^*\\ \mathbf{L}_{II}\overleftrightarrow{\mathbf{Y}}^*\end{pmatrix}=\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}$. Using properties of $\overleftrightarrow{\mathbf{Y}}^*$, we evaluate $E_R\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}\mathbf{Y}_I\\ \mathbf{Y}_{II}\end{pmatrix}$ and $E_{\xi R}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}\mathbf{X}_I\\ \mathbf{X}_{II}\end{pmatrix}\boldsymbol{\alpha}$ (where $\boldsymbol{\alpha}$ represents some parameters) and $\text{var}_{\xi R}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\begin{pmatrix}\mathbf{V}_I^* & \mathbf{V}_{I,II}^*\\ \mathbf{V}_{II,I}^* & \mathbf{V}_{II}^*\end{pmatrix}$. We illustrate this for collapsing a-g in Section 6. In evaluating these expressions, we use $E_{\xi R}\left(\overleftrightarrow{\mathbf{Y}}^*\right)=\left[\frac{1}{N}\left(\mathbf{1}_N\otimes\mathbf{I}_N\right)\right]\mathbf{y}$ and

$\text{var}_{\xi R}\left(\overleftrightarrow{\mathbf{Y}}^*\right)=\frac{1}{N}\mathbf{I}_N\otimes\left(\bigoplus_{s=1}^{N}\sigma_{se}^2\right)+\mathbf{P}_N\otimes\boldsymbol{\Delta}$.

a. Collapsing to the population frame $\begin{pmatrix}\mathbf{Y}_{RI}\\ \mathbf{Y}_{RII}\end{pmatrix}$ where $\mathbf{Y}_{RI}=\left(\left(Y_{sk}\right)\right)=\begin{pmatrix}Y_{1k} & Y_{2k} & \cdots & Y_{nk}\end{pmatrix}'$ and

$\mathbf{Y}_{RII}=\left(\left(Y_{sk}\right)\right)=\begin{pmatrix}Y_{(n+1)k} & Y_{(n+1)k} & \cdots & Y_{Nk}\end{pmatrix}'$: $\mathbf{L}=\mathbf{1}_N'\otimes\mathbf{I}_N$ and $\begin{pmatrix}\mathbf{L}_I\\ \mathbf{L}_{II}\end{pmatrix}=\left(\begin{array}{c}\mathbf{1}_N'\otimes\begin{pmatrix}\mathbf{I}_n & \mathbf{0}_{n\times N-n}\end{pmatrix}\\ \hline \mathbf{1}_N'\otimes\begin{pmatrix}\mathbf{0}_{(N-n)xn} & \mathbf{I}_{N-n}\end{pmatrix}\end{array}\right)$.

Then $E_{\xi R}\begin{pmatrix}\mathbf{Y}_{RI}\\ \mathbf{Y}_{RII}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}\mathbf{I}_n & \mathbf{0}_{n\times N-n}\end{pmatrix}\\ \begin{pmatrix}\mathbf{0}_{(N-n)xn} & \mathbf{I}_{N-n}\end{pmatrix}\end{pmatrix}\mathbf{y}$ and $\text{var}_{\xi R}\begin{pmatrix}\mathbf{Y}_{RI}\\ \mathbf{Y}_{RII}\end{pmatrix}=\left(\begin{array}{c|c}\bigoplus_{s=1}^{n}\sigma_{se}^2 & \mathbf{0}\\ \hline \mathbf{0} & \bigoplus_{s=n+1}^{N}\sigma_{se}^2\end{array}\right)$.

b. Usual random variables $\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}$ where $\mathbf{Y}_I^*=\left(\left(Y_{ik}^*\right)\right)=\begin{pmatrix}Y_{1k}^* & Y_{2k}^* & \cdots & Y_{nk}^*\end{pmatrix}'$ and

$\mathbf{Y}_{II}^*=\left(\left(Y_{ik}^*\right)\right)=\begin{pmatrix}Y_{(n+1)k}^* & Y_{(n+2)k}^* & \cdots & Y_{Nk}^*\end{pmatrix}'$: $\mathbf{L}=\mathbf{I}_N\otimes\mathbf{1}_N'$ and $\begin{pmatrix}\mathbf{L}_I\\ \mathbf{L}_{II}\end{pmatrix}=\left(\begin{array}{c}\begin{pmatrix}\mathbf{I}_n & \mathbf{0}_{n\times N-n}\end{pmatrix}\otimes\mathbf{1}_N'\\ \hline \begin{pmatrix}\mathbf{0}_{(N-n)xn} & \mathbf{I}_{N-n}\end{pmatrix}\otimes\mathbf{1}_N'\end{array}\right)$.

Then, $E_{\xi R}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\left(\begin{array}{c}\mathbf{1}_n\\ \hline \mathbf{1}_{N-n}\end{array}\right)\mu$ and $\text{var}_{\xi R}\begin{pmatrix}\mathbf{Y}_I^*\\ \mathbf{Y}_{II}^*\end{pmatrix}=\left(\begin{array}{c|c}\mathbf{I}_n-\frac{1}{N}\mathbf{J}_n & -\frac{1}{N}\mathbf{J}_{n\times(N-n)}\\ \hline -\frac{1}{N}\mathbf{J}_{(N-n)\times n} & \mathbf{I}_{N-n}-\frac{1}{N}\mathbf{J}_{N-n}\end{array}\right)\sigma^2+\left(\begin{array}{c|c}\mathbf{I}_n & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{I}_{N-n}\end{array}\right)\sigma_e^2$

where $\mathbf{1}_N'\boldsymbol{\Delta}\mathbf{1}_N=\sigma^2$. Hence $\mathbf{V}_I^*=\mathbf{I}_n\left(\sigma^2+\sigma_e^2\right)-\frac{1}{N}\mathbf{J}_n\sigma^2$.

c. Collapsing to sample and remainder totals $\begin{pmatrix} Y_I^* \\ Y_{II}^* \end{pmatrix}$ where $Y_I^* = \sum_{i=1}^{n} Y_{ik}^*$ and $Y_{II}^* = \sum_{i=n+1}^{N} Y_{ik}^*$:

$\mathbf{L} = \begin{pmatrix} \mathbf{1}_n' & \mathbf{0}_{N-n}' \\ \mathbf{0}_n' & \mathbf{1}_{N-n}' \end{pmatrix} \otimes \mathbf{1}_N'$ and $\begin{pmatrix} \mathbf{L}_I \\ \mathbf{L}_{II} \end{pmatrix} = \begin{pmatrix} \left(\mathbf{1}_n' \quad \mathbf{0}_{N-n}'\right) \otimes \mathbf{1}_N' \\ \hdashline \left(\mathbf{0}_n' \quad \mathbf{1}_{N-n}'\right) \otimes \mathbf{1}_N' \end{pmatrix}$. Then $E_{\xi R} \begin{pmatrix} Y_I^* \\ Y_{II}^* \end{pmatrix} = N \begin{pmatrix} f \\ \hdashline 1-f \end{pmatrix} \mu$ and $\text{var}_{\xi R} \begin{pmatrix} Y_I^* \\ Y_{II}^* \end{pmatrix} = Nf(1-f) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sigma^2 + N \begin{pmatrix} f & 0 \\ 0 & 1-f \end{pmatrix} \sigma_e^2$.

d. Collapsing to sample and remainder means $\begin{pmatrix} \bar{Y}_I^* \\ \bar{Y}_{II}^* \end{pmatrix}$ where $\bar{Y}_I^* = \frac{1}{n} \sum_{i=1}^{n} Y_{ik}^*$ and $\bar{Y}_{II}^* = \frac{1}{N-n} \sum_{i=n+1}^{N} Y_{ik}^*$:

$\mathbf{L} = \frac{1}{N} \begin{pmatrix} \frac{1}{f}\mathbf{1}_n' & \mathbf{0}_{N-n}' \\ \mathbf{0}_n' & \frac{1}{1-f}\mathbf{1}_{N-n}' \end{pmatrix} \otimes \mathbf{1}_N'$ and $\begin{pmatrix} \mathbf{L}_I \\ \mathbf{L}_{II} \end{pmatrix} = \begin{pmatrix} \frac{1}{N}\left(\frac{1}{f}\mathbf{1}_n' \quad \mathbf{0}_{N-n}'\right) \otimes \mathbf{1}_N' \\ \hdashline \frac{1}{N}\left(\mathbf{0}_n' \quad \frac{1}{1-f}\mathbf{1}_{N-n}'\right) \otimes \mathbf{1}_N' \end{pmatrix}$. Then

$E_{\xi R} \begin{pmatrix} \bar{Y}_I^* \\ \bar{Y}_{II}^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mu$ and $\text{var}_{\xi R} \begin{pmatrix} \bar{Y}_I^* \\ \bar{Y}_{II}^* \end{pmatrix} = \frac{1}{n(1-f)} \begin{pmatrix} (1-f)^2 & -f(1-f) \\ -f(1-f) & f^2 \end{pmatrix} \sigma^2 + \frac{1}{N} \begin{pmatrix} \frac{1}{f} & 0 \\ 0 & \frac{1}{1-f} \end{pmatrix} \sigma_e^2$.

e. Collapsing to weighted population frame: $\begin{pmatrix} \mathbf{Y}_{wRI} \\ \mathbf{Y}_{wRII} \end{pmatrix}$ where

$\mathbf{Y}_{wRI} = \left(\left(Y_{wsk}\right)\right) = \left(Y_{w1k} \quad Y_{wsk} \quad \cdots \quad Y_{wnk}\right)'$ and $\mathbf{Y}_{wRII} = \left(\left(Y_{wsk}\right)\right) = \left(Y_{w(n+1)k} \quad Y_{w(n+2)k} \quad \cdots \quad Y_{wNk}\right)'$:

$\mathbf{L} = \mathbf{1}_N' \otimes \left(\bigoplus_{s=1}^{N} w_s\right)$ where $\begin{pmatrix} \mathbf{L}_I \\ \mathbf{L}_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_N' \otimes \left(\bigoplus_{s=1}^{n} w_s \quad \underset{n \times N-n}{\mathbf{0}}\right) \\ \hdashline \mathbf{1}_N' \otimes \left(\underset{(N-n)xn}{\mathbf{0}} \quad \bigoplus_{s=n+1}^{N} w_s\right) \end{pmatrix}$. Then

$E_{\xi R} \begin{pmatrix} \mathbf{Y}_{wRI} \\ \mathbf{Y}_{wRII} \end{pmatrix} = \begin{pmatrix} \left(\bigoplus_{s=1}^{n} w_s \quad \underset{n \times N-n}{\mathbf{0}}\right) \\ \hdashline \left(\underset{(N-n)xn}{\mathbf{0}} \quad \bigoplus_{s=n+1}^{N} w_s\right) \end{pmatrix} \begin{pmatrix} \mathbf{y}_I \\ \mathbf{y}_{II} \end{pmatrix}$ where $\mathbf{y}_I = \left(\left(y_s\right)\right) = \left(y_1 \quad y_2 \quad \cdots \quad y_n\right)'$ and

$\mathbf{y}_{II} = \left(\left(y_s\right)\right) = \left(y_{n+1} \quad y_{n+2} \quad \cdots \quad y_N\right)'$. Also, $\text{var}_{\xi R} \begin{pmatrix} \mathbf{Y}_{wRI} \\ \mathbf{Y}_{wRII} \end{pmatrix} = \begin{pmatrix} \bigoplus_{s=1}^{n} w_s^2 \sigma_{se}^2 & \underset{n \times N-n}{\mathbf{0}} \\ \underset{(N-n)xn}{\mathbf{0}} & \bigoplus_{s=n+1}^{N} w_s^2 \sigma_{se}^2 \end{pmatrix}$.

f. Collapsing to weighted population random variables: $\begin{pmatrix}\mathbf{Y}_{wI}^{*}\\ \mathbf{Y}_{wII}^{*}\end{pmatrix}$ where

$\mathbf{Y}_{wI}^{*}=\left(\left(Y_{wik}^{*}\right)\right)=\begin{pmatrix}Y_{w1k}^{*} & Y_{w2k}^{*} & \cdots & Y_{wnk}^{*}\end{pmatrix}'$ and $\mathbf{Y}_{wII}^{*}=\left(\left(Y_{wik}^{*}\right)\right)=\begin{pmatrix}Y_{w(n+1)k}^{*} & Y_{w(n+2)k}^{*} & \cdots & Y_{wNk}^{*}\end{pmatrix}'$:

$\mathbf{L}=\mathbf{1}_N'\otimes\mathbf{w}'$ where $\begin{pmatrix}\mathbf{L}_I\\ \mathbf{L}_{II}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}\mathbf{I}_n & \underset{n\times N-n}{\mathbf{0}}\end{pmatrix}\otimes\mathbf{w}'\\ \hline \begin{pmatrix}\underset{(N-n)xn}{\mathbf{0}} & \mathbf{I}_{N-n}\end{pmatrix}\otimes\mathbf{w}'\end{pmatrix}$. Then $E_{\xi R}\begin{pmatrix}\mathbf{Y}_{wI}^{*}\\ \mathbf{Y}_{wII}^{*}\end{pmatrix}=\begin{pmatrix}\mathbf{1}_n\\ \mathbf{1}_{N-n}\end{pmatrix}\mu_w$ where we define $\mu_w=\frac{1}{N}\sum_{s=1}^{N}w_s y_s$. Also,

$$\text{var}_{\xi R}\begin{pmatrix}\mathbf{Y}_{wI}^{*}\\ \mathbf{Y}_{wII}^{*}\end{pmatrix}=\left(\begin{array}{c|c}\mathbf{I}_n-\frac{1}{N}\mathbf{J}_n & -\frac{1}{N}\mathbf{J}_{n\times(N-n)}\\ \hline -\frac{1}{N}\mathbf{J}_{(N-n)\times n} & \mathbf{I}_{N-n}-\frac{1}{N}\mathbf{J}_{N-n}\end{array}\right)\sigma_w^2+\left(\begin{array}{c|c}\mathbf{I}_n & \mathbf{0}_{n\times N-n}\\ \hline \mathbf{0}_{(N-n)xn} & \mathbf{I}_{N-n}\end{array}\right)\sigma_{we}^2$$

where $\sigma_w^2=\mathbf{w}'\boldsymbol{\Delta}\mathbf{w}$ or $\sigma_w^2=\frac{1}{N-1}\sum_{s=1}^{N}\left(w_s y_s-\mu_w\right)^2$, and $\sigma_{we}^2=\frac{1}{N}\sum_{s=1}^{N}w_s^2\sigma_{se}^2=\frac{1}{N}\mathbf{w}'\left(\bigoplus_{s=1}^{N}\sigma_{se}^2\right)\mathbf{w}$.

g. Collapsing to weighted sample and remainder totals $\begin{pmatrix}Y_{wI}^{*}\\ Y_{wII}^{*}\end{pmatrix}$ where $Y_{wI}^{*}=\sum_{i=1}^{n}Y_{wik}^{*}$ and

$Y_{wII}^{*}=\sum_{i=n+1}^{N}Y_{wik}^{*}$: $\mathbf{L}=\begin{pmatrix}\mathbf{1}_n' & \mathbf{0}_{N-n}'\\ \mathbf{0}_n' & \mathbf{1}_{N-n}'\end{pmatrix}\otimes\mathbf{w}'$ where $\begin{pmatrix}\mathbf{L}_I\\ \mathbf{L}_{II}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}\mathbf{1}_n' & \mathbf{0}_{N-n}'\end{pmatrix}\otimes\mathbf{w}'\\ \hline \begin{pmatrix}\mathbf{0}_n' & \mathbf{1}_{N-n}'\end{pmatrix}\otimes\mathbf{w}'\end{pmatrix}$. Then

$E_{\xi R}\begin{pmatrix}Y_{wI}^{*}\\ Y_{wII}^{*}\end{pmatrix}=N\begin{pmatrix}f\\ 1-f\end{pmatrix}\mu_w$ and $\text{var}_{\xi R}\begin{pmatrix}Y_{wI}^{*}\\ Y_{wII}^{*}\end{pmatrix}=Nf\left(1-f\right)\left(\begin{array}{c|c}1 & -1\\ \hline -1 & 1\end{array}\right)\sigma_w^2+N\left(\begin{array}{c|c}f & 0\\ \hline 0 & 1-f\end{array}\right)\sigma_{we}^2$.

10. TARGET PARAMETERS, RANDOM VARIABLES, AND TERMINOLOGY

We assume that there is an interest in a ‘target’ that can be defined as a linear combination of the expected value (over measurement error) of the collapsed random variables. We use the sample data to estimate/predict the target.

If the target is a fixed constant, then we use the sample data to “estimate” the target.

If the target is a random variable, then we use the sample data to “predict” the target.

We only consider inference for a single ‘target’ (not joint targets).

We represent collapsed random variables by $\mathbf{L}\overleftrightarrow{\mathbf{Y}}^{*}=\begin{pmatrix}\mathbf{Y}_I^{*}\\ \mathbf{Y}_{II}^{*}\end{pmatrix}$. We define a ‘target’ as $P=\mathbf{g}'\begin{pmatrix}\mathbf{Y}_I\\ \mathbf{Y}_{II}\end{pmatrix}$

where $E_R\begin{pmatrix}\mathbf{Y}_I^{*}\\ \mathbf{Y}_{II}^{*}\end{pmatrix}=\begin{pmatrix}\mathbf{Y}_I\\ \mathbf{Y}_{II}\end{pmatrix}$ and partition $\mathbf{g}$ representing $\mathbf{g}'=\begin{pmatrix}\mathbf{g}_I' & \mathbf{g}_{II}'\end{pmatrix}$ so that $P=\mathbf{g}_I'\mathbf{Y}_I+\mathbf{g}_{II}'\mathbf{Y}_{II}$.

Example: Suppose we collapse random variables to sample and remainder means $\begin{pmatrix} \bar{Y}_I^* \\ \bar{Y}_{II}^* \end{pmatrix}$ where $E_R\begin{pmatrix} \bar{Y}_I^* \\ \bar{Y}_{II}^* \end{pmatrix} = \begin{pmatrix} \bar{Y}_I \\ \bar{Y}_{II} \end{pmatrix}$. Assume we are interested in the target $P = \mu$, a fixed constant. Since $P = f\bar{Y}_I + (1-f)\bar{Y}_{II}$, $\mathbf{g}' = \begin{pmatrix} f & (1-f) \end{pmatrix}$ where $g_I = f$ and $g_{II} = 1-f$.

11. DEVELOPING THE BEST LINEAR UNBIASED PREDICTOR (BLUP)

We define the BLUP of P as $\hat{P}$, where $\hat{P}$ satisfies the following criteria (see Royall, 1976):

Linear in the sample: $\hat{P} = (\mathbf{g}_I' + \mathbf{a}')\mathbf{Y}_I^*$
Unbiased: $E_{\xi R}\left(\hat{P} - P\right) = 0$
Minimum MSE: $\text{var}_{\xi R}\left(\hat{P} - P\right)$ is minimized.

In order to develop the BLUP of P, we first present expressions for $\hat{P} - P$ and its variance. First, note that

$$\hat{P} - P = \left((\mathbf{g}_I' + \mathbf{a}') \;\vdots\; -\mathbf{g}_I' \;\vdots\; -\mathbf{g}_{II}' \right) \begin{pmatrix} \mathbf{Y}_I^* \\ \hline \mathbf{Y}_I \\ \hline \mathbf{Y}_{II} \end{pmatrix}.$$

We represent $\text{var}_\xi \begin{pmatrix} \mathbf{Y}_I \\ \mathbf{Y}_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II} \end{pmatrix}$. Then

$$\text{var}_{\xi R}\begin{pmatrix} \mathbf{Y}_I^* \\ \hline \mathbf{Y}_I \\ \hline \mathbf{Y}_{II} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_I^* & \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_I & \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II,I} & \mathbf{V}_{II} \end{pmatrix}.$$

Let us define $\mathbf{V}_I^* = \mathbf{V}_I + \mathbf{V}_{RI}^*$. Then

$$\text{var}_{\xi R}\begin{pmatrix} \mathbf{Y}_I^* \\ \hline \mathbf{Y}_I \\ \hline \mathbf{Y}_{II} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathbf{V}_{RI}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The Unbiased Constraint:

We can expand the unbiased constraint: $E\left(\hat{P} - P\right) = \begin{pmatrix} \mathbf{a}' & -\mathbf{g}_{II}' \end{pmatrix} \begin{pmatrix} \mathbf{X}_I \\ \mathbf{X}_{II} \end{pmatrix} \boldsymbol{\alpha}$, such that $E\left(\hat{P} - P\right) = (\mathbf{a}'\mathbf{X}_I - \mathbf{g}_{II}'\mathbf{X}_{II})\boldsymbol{\alpha}$. In order for this expression to equal zero for any value of $\boldsymbol{\alpha}$, the unbiased constraint will be always be satisfied when $\mathbf{a}'\mathbf{X}_I - \mathbf{g}'_{II}\mathbf{X}_{II} = \mathbf{0}$. This is introduced as a constraint using Lagrangian multipliers when minimizing the MSE.

<u>Finding the Minimum MSE</u>:

Notice that

$$\text{var}_{\xi R}\left(\hat{P}-P\right) = \left(\left(\mathbf{g}'_I + \mathbf{a}'\right) \mid -\mathbf{g}'_I \mid -\mathbf{g}'_{II}\right)\begin{pmatrix}\mathbf{V}_I^* & \mathbf{V}_I & \mathbf{V}_{I,II}\\ \mathbf{V}_I & \mathbf{V}_I & \mathbf{V}_{I,II}\\ \mathbf{V}_{II,I} & \mathbf{V}_{II,I} & \mathbf{V}_{II}\end{pmatrix}\begin{pmatrix}\mathbf{g}_I+\mathbf{a}\\ \hline -\mathbf{g}_I \\ \hline -\mathbf{g}_{II}\end{pmatrix}, \text{ or}$$

$$\text{var}_{\xi R}\left(\hat{P}-P\right) = \begin{pmatrix}\mathbf{a}' & -\mathbf{g}'_{II}\end{pmatrix}\begin{pmatrix}\mathbf{V}_I & \mathbf{V}_{I,II}\\ \mathbf{V}_{II,I} & \mathbf{V}_{II}\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ -\mathbf{g}_{II}\end{pmatrix} + \left(\mathbf{g}'_I + \mathbf{a}'\right)\mathbf{V}_{RI}^*\left(\mathbf{g}_I + \mathbf{a}\right).$$

Expanding this expression, $\text{var}\left(\hat{P}-P\right) = \mathbf{a}'\mathbf{V}_I^*\mathbf{a} + 2\left(\mathbf{g}'_I\mathbf{V}_{RI}^* - \mathbf{g}'_{II}\mathbf{V}_{II,I}\right)\mathbf{a} + \mathbf{g}'_{II}\mathbf{V}_{II}\mathbf{g}_{II} + \mathbf{g}'_I\mathbf{V}_{RI}^*\mathbf{g}_I$.

Including the constraint via a Lagrangian multiplier, we seek to find the value of $\mathbf{a}$ that will minimize

$$\begin{aligned} f\left(\mathbf{a},\boldsymbol{\lambda}\right) = \mathbf{a}'\mathbf{V}_I^*\mathbf{a} + 2\left(\mathbf{g}'_I\mathbf{V}_{RI}^* - \mathbf{g}'_{II}\mathbf{V}_{II,I}\right)\mathbf{a} + 2\left(\mathbf{a}'\mathbf{X}_I - \mathbf{g}'_{II}\mathbf{X}_{II}\right)\boldsymbol{\lambda} \\ +\mathbf{g}'_{II}\mathbf{V}_{II}\mathbf{g}_{II} + \mathbf{g}'_I\mathbf{V}_{RI}^*\mathbf{g}_I \end{aligned}.$$

Differentiating with respect to $\mathbf{a}$ and $\boldsymbol{\lambda}$,

$$\frac{\partial f\left(\mathbf{a},\boldsymbol{\lambda}\right)}{\partial \mathbf{a}} = 2\mathbf{V}_I^*\mathbf{a} + 2\left(\mathbf{V}_{RI}^*\mathbf{g}_I - \mathbf{V}_{I,II}\mathbf{g}_{II}\right) + 2\mathbf{X}_I\boldsymbol{\lambda} \text{ and}$$

$$\frac{\partial f\left(\mathbf{a},\boldsymbol{\lambda}\right)}{\partial \boldsymbol{\lambda}} = 2\left(\mathbf{X}'_I\mathbf{a} - \mathbf{X}'_{II}\mathbf{g}_{II}\right).$$

To find the value of $\mathbf{a}$ that minimizes $f\left(\mathbf{a},\boldsymbol{\lambda}\right)$, we set these derivatives to zero simultaneously, and solve for $\mathbf{a}$. The estimating equations are given by

$$\frac{1}{2}\begin{pmatrix}\dfrac{\partial f\left(\hat{\mathbf{a}},\hat{\boldsymbol{\lambda}}\right)}{\partial \mathbf{a}}\\ \dfrac{\partial f\left(\hat{\mathbf{a}},\hat{\boldsymbol{\lambda}}\right)}{\partial \boldsymbol{\lambda}}\end{pmatrix} = \begin{pmatrix}\mathbf{V}_I^* & \mathbf{X}_I\\ \mathbf{X}'_I & \mathbf{0}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{a}}\\ \hat{\boldsymbol{\lambda}}\end{pmatrix} - \begin{pmatrix}-\mathbf{V}_{RI}^*\mathbf{g}_I + \mathbf{V}_{I,II}\mathbf{g}_{II}\\ \mathbf{X}'_{II}\mathbf{g}_{II}\end{pmatrix} = \begin{pmatrix}\mathbf{0}_n\\ \mathbf{0}\end{pmatrix}$$

or equivalently, by $\begin{pmatrix}\mathbf{V}_I^* & \mathbf{X}_I\\ \mathbf{X}_I' & \mathbf{0}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{a}}\\ \hat{\boldsymbol{\lambda}}\end{pmatrix} = \begin{pmatrix}-\mathbf{V}_{RI}^*\mathbf{g}_I + \mathbf{V}_{I,II}\mathbf{g}_{II}\\ \mathbf{X}'_{II}\mathbf{g}_{II}\end{pmatrix}$.

To solve this equation for $\hat{\mathbf{a}}$, we make use of results on the inverse of a partitioned matrix $\mathbf{X} = \left(\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \mathbf{C} & \mathbf{D}\end{array}\right)$, such that $\mathbf{X}^{-1} = \left(\begin{array}{c|c}\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{Q}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{Q}^{-1}\\ \hline -\mathbf{Q}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{Q}^{-1}\end{array}\right)$ where $\mathbf{Q} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$. As a result,

$$\begin{pmatrix} \mathbf{V}_I^* & \mathbf{X}_I \\ \mathbf{X}_I' & \mathbf{0} \end{pmatrix}^{-1} = \left(\begin{array}{c|c} \mathbf{V}_I^{*-1} - \mathbf{V}_I^{*-1}\mathbf{X}_I\left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{*-1} & \mathbf{V}_I^{*-1}\mathbf{X}_I\left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1} \\ \hline \left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{*-1} & -\left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1} \end{array} \right),$$

$$\hat{\mathbf{a}} = \left[\mathbf{V}_I^{*-1} - \mathbf{V}_I^{*-1}\mathbf{X}_I\left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{*-1}\right]\left(-\mathbf{V}_{RI}^*\mathbf{g}_I + \mathbf{V}_{I,II}\mathbf{g}_{II}\right) + \mathbf{V}_I^{*-1}\mathbf{X}_I\left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_{II}'\mathbf{g}_{II}.$$

We can now express the best linear unbiased predictor. Recall that $P = \mathbf{g}_I'\mathbf{Y}_I + \mathbf{g}_{II}'\mathbf{Y}_{II}$, and we predict P by $\hat{P} = \left(\mathbf{g}_I' + \mathbf{a}'\right)\mathbf{Y}_I^*$, where the best predictor replaces $\mathbf{a}$ by $\hat{\mathbf{a}}$. Let us define $\hat{\alpha} = \left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{Y}_I^*$. Then

$$\begin{aligned} \hat{\mathbf{a}}'\mathbf{Y}_I^* &= \left(-\mathbf{g}_I'\mathbf{V}_{RI}^* + \mathbf{g}_{II}'\mathbf{V}_{II,I}\right)\left[\mathbf{V}_I^{*-1}\mathbf{Y}_I^* - \mathbf{V}_I^{*-1}\mathbf{X}_I\left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{Y}_I^*\right] + \mathbf{g}_{II}'\mathbf{X}_{II}\left(\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I'\mathbf{V}_I^{*-1}\mathbf{Y}_I^* \\ &= -\mathbf{g}_I'\mathbf{V}_{RI}^*\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right) + \mathbf{g}_{II}'\left[\mathbf{X}_{II}\hat{\alpha} + \mathbf{V}_{II,I}\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] \end{aligned}$$

.

As a result, the best linear unbiased predictor is given by

$$\hat{P} = \mathbf{g}_I'\left[\mathbf{Y}_I^* - \mathbf{V}_{RI}^*\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] + \mathbf{g}_{II}'\left[\mathbf{X}_{II}\hat{\alpha} + \mathbf{V}_{II,I}\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right].$$

We can express the predictor in a slightly different manner by substituting $\mathbf{V}_{RI}^* = \mathbf{V}_I^* - \mathbf{V}_I$, resulting in

$$\hat{P} = \mathbf{g}_I'\left[\mathbf{X}_I\hat{\alpha} + \mathbf{V}_I\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] + \mathbf{g}_{II}'\left[\mathbf{X}_{II}\hat{\alpha} + \mathbf{V}_{II,I}\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right].$$

<u>Example 1.</u>

Target: $\mu = \frac{1}{N}\sum_{i=1}^{N} Y_i$

Collapsing: $\mathbf{L}\vec{\mathbf{Y}}^* = \mathbf{Y}^*$ (usual random variables)

$E\begin{pmatrix} \mathbf{Y}_I^* \\ \mathbf{Y}_{II}^* \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n \\ \hline \mathbf{1}_{N-n} \end{pmatrix}\mu$ and $\begin{pmatrix} \mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II} \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{I}_n - \frac{1}{N}\mathbf{J}_n & -\frac{1}{N}\mathbf{J}_{n\times(N-n)} \\ \hline -\frac{1}{N}\mathbf{J}_{(N-n)\times n} & \mathbf{I}_{N-n} - \frac{1}{N}\mathbf{J}_{N-n} \end{array} \right)\sigma^2$ and

$\mathbf{V}_I^* = \left(\sigma^2 + \sigma_e^2\right)\left(\mathbf{I}_n - \frac{\sigma^2}{\left(\sigma^2 + \sigma_e^2\right)N}\mathbf{J}_n\right)$. Now $\begin{pmatrix} \mathbf{g}_I' & \mathbf{g}_{II}' \end{pmatrix} = \begin{pmatrix} \frac{1}{N}\mathbf{1}_n' & \frac{1}{N}\mathbf{1}_{N-n}' \end{pmatrix}$ so that the unbiased constraint simplifies to $\mathbf{a}'\mathbf{1}_n = 1 - f$. To evaluate the predictor, we note that $\left(\mathbf{R} + \mathbf{s}\mathbf{u}'\right)^{-1} = \mathbf{R}^{-1} - \left(1 + \mathbf{u}'\mathbf{R}^{-1}\mathbf{s}\right)^{-1}\mathbf{R}^{-1}\mathbf{s}\mathbf{u}'\mathbf{R}^{-1}$. As a special case, $\left(\mathbf{I}_n - \frac{1}{N}\mathbf{J}_n\right)^{-1} = \mathbf{I}_n + \frac{1}{N-n}\mathbf{J}_n$. As a result,

$$\mathbf{V}_I^{*-1} = \frac{1}{\sigma^2+\sigma_e^2}\left(\mathbf{I}_n + \frac{1}{\left(\frac{\sigma^2+\sigma_e^2}{\sigma^2}\right)N-n}\mathbf{J}_n\right) \quad \text{or}$$

$$= \frac{1}{\sigma^2+\sigma_e^2}\left(\mathbf{I}_n + \frac{\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\mathbf{J}_n\right)$$

$$\mathbf{V}_I^{*-1} = \frac{1}{\sigma^2+\sigma_e^2}\left(\mathbf{I}_n + \frac{\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\mathbf{J}_n\right).$$

Other terms simplify such that

$$\hat{\alpha} = \left(\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{X}_I\right)^{-1}\mathbf{X}_I{}'\mathbf{V}_I^{*-1}\mathbf{Y}_I^* = \frac{1}{n}\sum_{i=1}^n Y_i^* = \bar{Y}_I^*;$$

$$\begin{aligned}
\mathbf{V}_I\mathbf{V}_I^{*-1} &= \frac{\sigma^2}{\sigma^2+\sigma_e^2}\left(\mathbf{I}_n - \frac{1}{N}\mathbf{J}_n\right)\left(\mathbf{I}_n + \frac{\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\mathbf{J}_n\right)\\
&= \frac{\sigma^2}{\sigma^2+\sigma_e^2}\left(\mathbf{I}_n - \frac{1}{N}\mathbf{J}_n + \frac{(1-f)N\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\frac{1}{N}\mathbf{J}_n\right)\\
&= \frac{\sigma^2}{\sigma^2+\sigma_e^2}\left(\mathbf{I}_n + \frac{-\sigma^2N-\sigma_e^2N+n\sigma^2+N\sigma^2-n\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\frac{1}{N}\mathbf{J}_n\right)\\
&= \frac{\sigma^2}{\sigma^2+\sigma_e^2}\left(\mathbf{I}_n + \frac{-\sigma_e^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\mathbf{J}_n\right)
\end{aligned};$$

and

$$\begin{aligned}
\mathbf{V}_{II,I}\mathbf{V}_I^{*-1} &= -\frac{1}{N}\frac{\sigma^2}{\sigma^2+\sigma_e^2}\mathbf{1}_{N-n}\mathbf{1}_n'\left(\mathbf{I}_n + \frac{\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\mathbf{J}_n\right)\\
&= -\frac{1}{N}\frac{\sigma^2}{\sigma^2+\sigma_e^2}\mathbf{1}_{N-n}\mathbf{1}_n'\left(1 + \frac{n\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\right)\\
&= -\frac{1}{N}\left(\frac{\sigma^2}{\sigma^2+\sigma_e^2}\right)\frac{\left(\sigma^2+\sigma_e^2\right)N}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\mathbf{1}_{N-n}\mathbf{1}_n'\\
&= -\frac{\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N-n\sigma^2}\mathbf{1}_{N-n}\mathbf{1}_n'
\end{aligned}$$

so that $\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha} = \mathbf{P}_n\mathbf{Y}_I^*$. Using these terms, $\mathbf{V}_I\mathbf{V}_I^{*-1}\mathbf{P}_n = \frac{\sigma^2}{\sigma^2+\sigma_e^2}\mathbf{P}_n$ and $\mathbf{V}_{II,I}\mathbf{V}_I^{*-1}\mathbf{P}_n = \mathbf{0}$. As a result,

$$\begin{aligned}\hat{P} &= \mathbf{g}_I{}'\left[\mathbf{X}_I\hat{\alpha} + \mathbf{V}_I\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] + \mathbf{g}_{II}'\left[\mathbf{X}_{II}\hat{\alpha} + \mathbf{V}_{II,I}\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] \\ &= \mathbf{g}_I{}'\left[\mathbf{X}_I\hat{\alpha} + \frac{\sigma^2}{\sigma^2+\sigma_e^2}\mathbf{P}_n\mathbf{Y}_I^*\right] + \mathbf{g}_{II}'\left[\mathbf{X}_{II}\hat{\alpha}\right] \\ &= \frac{1}{N}\mathbf{1}_n'\left[\mathbf{1}_n\bar{Y}_I^* + \frac{\sigma^2}{\sigma^2+\sigma_e^2}\mathbf{P}_n\mathbf{Y}_I^*\right] + \frac{1}{N}\mathbf{1}_{N-n}'\mathbf{1}_{N-n}\bar{Y}_I^* \\ &= \bar{Y}_I^*\end{aligned} .$$

Example 2.

Target: $\mu = f\bar{Y}_I + (1-f)\bar{Y}_{II}$

Collapse random variables to sample and remainder means $\begin{pmatrix}\bar{Y}_I^* \\ \bar{Y}_{II}^*\end{pmatrix}$

$E\begin{pmatrix}\bar{Y}_I^* \\ \bar{Y}_{II}^*\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\mu$ and $\text{var}\begin{pmatrix}\bar{Y}_I \\ \bar{Y}_{II}\end{pmatrix} = \frac{1}{n(1-f)}\left(\begin{array}{c|c}(1-f)^2 & -f(1-f) \\ \hline -f(1-f) & f^2\end{array}\right)\sigma^2$ and

$\mathbf{V}_I^* = \frac{1-f}{n}\sigma^2 + \frac{1}{fn}\sigma_e^2$.

Now $\left(\mathbf{g}_I' \quad \mathbf{g}_{II}'\right) = \left(f \quad (1-f)\right)$ so that the unbiased constraint simplifies to $a = 1-f$. There is no other choice for a that will be unbiased, so we set $\hat{a} = 1-f$. As a result, the BLUP is given by

$$\begin{aligned}\hat{P} &= \left(\mathbf{g}_I' + \hat{\mathbf{a}}'\right)\mathbf{Y}_I^* \\ &= \left(f + (1-f)\right)\bar{Y}_I^* . \\ &= \bar{Y}_I^*\end{aligned}$$

Example 3.

Target: Y_i

Collapsing: $\mathbf{L}\vec{\mathbf{Y}}^* = \mathbf{Y}^*$ (usual random variables)

$$E\begin{pmatrix}\mathbf{Y}_I^* \\ \mathbf{Y}_{II}^*\end{pmatrix} = \begin{pmatrix}\mathbf{1}_n \\ \hline \mathbf{1}_{N-n}\end{pmatrix}\mu \text{ and } \begin{pmatrix}\mathbf{V}_I & \mathbf{V}_{I,II} \\ \mathbf{V}_{II,I} & \mathbf{V}_{II}\end{pmatrix} = \left(\begin{array}{c|c}\mathbf{I}_n - \frac{1}{N}\mathbf{J}_n & -\frac{1}{N}\mathbf{J}_{n\times(N-n)} \\ \hline -\frac{1}{N}\mathbf{J}_{(N-n)\times n} & \mathbf{I}_{N-n} - \frac{1}{N}\mathbf{J}_{N-n}\end{array}\right)\sigma^2 \text{ and}$$

$\mathbf{V}_I^* = \left(\sigma^2 + \sigma_e^2\right)\left(\mathbf{I}_n - \frac{\sigma^2}{\left(\sigma^2+\sigma_e^2\right)N}\mathbf{J}_n\right)$.

Now $\left(\mathbf{g}_I' \quad \mathbf{g}_{II}'\right) = \left(\mathbf{e}_{iI}' \quad \mathbf{e}_{iII}'\right)$, where $\mathbf{e}_i$ is an $N\times 1$ vector with element i equal to one, and all other elements equal to zero, partitioned to conform. The unbiased constraint simplifies to $\mathbf{a}'\mathbf{1}_n = \mathbf{e}_{iII}'\mathbf{1}_{N-n}$. Using the simplifications from Example 1,

$$\begin{aligned}\hat{P} &= \mathbf{g}_I{}'\left[\mathbf{X}_I\hat{\alpha} + \mathbf{V}_I\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] + \mathbf{g}'_{II}\left[\mathbf{X}_{II}\hat{\alpha} + \mathbf{V}_{II,I}\mathbf{V}_I^{*-1}\left(\mathbf{Y}_I^* - \mathbf{X}_I\hat{\alpha}\right)\right] \\ &= \mathbf{g}_I{}'\left[\mathbf{X}_I\hat{\alpha} + \frac{\sigma^2}{\sigma^2 + \sigma_e^2}\mathbf{P}_n\mathbf{Y}_I^*\right] + \mathbf{g}'_{II}\left[\mathbf{X}_{II}\hat{\alpha}\right] \\ &= \mathbf{g}_I{}'\left[\mathbf{1}_n\overline{Y}_I^* + \frac{\sigma^2}{\sigma^2 + \sigma_e^2}\mathbf{P}_n\mathbf{Y}_I^*\right] + \mathbf{g}'_{II}\left[\mathbf{1}_{N-n}\overline{Y}_I^*\right]\end{aligned}$$

Thus, when $i \le n$, $\hat{P} = \overline{Y}_I^* + \frac{\sigma^2}{\sigma^2 + \sigma_e^2}\left(Y_i^* - \overline{Y}_I^*\right)$. When $i > n$, $\hat{P} = \overline{Y}_I^*$.