Example of estimation in mixed logistic model
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Considering the mixed logistic model in the example of mixedrz05r03:
\[ x_{ij} \mid u_i \sim Bernouli(\pi_{ij}) \]

and
\[ \eta_{ij} = \log \left( \frac{\pi_{ij}}{1 - \pi_{ij}} \right) = \beta_0 + u_i, \]
\[ \pi_{ij} = \frac{\exp(\beta_0 + u_i)}{1 + \exp(\beta_0 + u_i)}, \]

where
\[ i = 1, \ldots, n, \quad j = 1, \ldots, m, \]
\[ x_{ij} \] denotes the corresponding favorable cures,
\[ u_i \] are assumed to be iid \( N(0, \sigma_u^2) \).
\[ \beta_0 \quad \text{and} \quad \beta_1 \] are regression coefficient.

The model can be rewritten as:
\[ \eta_{ij} = \log \left( \frac{p_{ij}}{1 - p_{ij}} \right) = t_j \beta + u_i, \]
\[ p_{ij} = \frac{\exp(t_j \beta + u_i)}{1 + \exp(t_j \beta + u_i)}, \]

where
\[ \beta = (\beta_0, \beta_1), \]
\[ t_j = (1, t_j). \]

Since \( p_{ij} = \Pr(x_{ij} = 1 \mid u_i) \), the logistic regression model is
\[ \log \left( \frac{\Pr(x_{ij} = 1 \mid u_i)}{1 + \Pr(x_{ij} = 1 \mid u_i)} \right) = t_j \beta + u_i. \]

In this model, we assume the observations within a clinic are independent given the random effect
\[ \Pr(x_{ij} = 1 \mid u_i) = p_{ij} \]

and
\[ \Pr(x_{ij} = 0 \mid u_i) = 1 - p_{ij}. \]

Then we can get the conditional likelihood for the observations in \( t^{th} \) clinic:
\[ l(X_i \mid u_i) = \prod_{j=1}^{m} p_{ij}^{x_{ij}} [1 - p_{ij}]^{1-x_{ij}} \]

We can get the likelihood of the \( m \) response patterns in all of clinics by integrating over the distribution of the random effects:
\[ h(X_i) = \int \theta l(X_i \mid \theta) f(\theta) \, d\theta, \]

where \( f(\theta) \) is the population distribution of the random effects, namely \( N(0, \sigma^2_u) \).

Thus, the total sample marginal likelihoods is

\[ L = \prod_{i=1}^{n} h(X_i) \]

and

\[ \log L = \sum_{i=1}^{n} \log h(X_i). \]

The maximum (marginal) likelihood can be obtained by differentiate the above log-likelihood function:

\[
\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{h(X_i)} \frac{\partial h(X_i)}{\partial \alpha} \\
= \sum_{i=1}^{n} \frac{1}{h(X_i)} \int_{\theta} \frac{\partial l(X_i \mid \theta)}{\partial \alpha} f(\theta) \, d\theta \\
= \sum_{i=1}^{n} \frac{1}{h(X_i)} \int_{\theta} \frac{\partial}{\partial \alpha} \prod_{j=1}^{m} \pi_{ij}^{x_{ij}} (1 - \pi_{ij})^{-x_{ij}} f(\theta) \, d\theta \\
= \sum_{i=1}^{n} \frac{1}{h(X_i)} \int_{\theta} \sum_{j=1}^{m} \left[ X_{ij} \frac{\partial \pi_{ij}}{\partial \alpha} + \frac{1 - X_{ij}}{p_{ij}} \frac{\partial (-\pi_{ij})}{\partial \alpha} \right] l(X_i \mid \theta) f(\theta) \, d\theta
\]

where \( \alpha = (\beta_0, \beta_1) \)

Various approximations for evaluating the integral over the random-effects distribution have been developed. The most frequently used methods are based on first or second-order Taylor expansions.