Predictors for Simple 1-Factor Completely Randomized Design with Permutation Treatments

Introduction

We sketch the development of estimators of treatment parameters in a simple 1 way completely randomized design using a random permutation model prediction framework. The context we consider is for a finite population with labeled units indexed by \( s = 1, \ldots, N \). We assume that treatments are indexed by \( a = 1, \ldots, A \). The experiment consists of randomly assigning a simple random sample of \( n \) subjects to each of the \( A \) treatments (where \( nA \leq N \)).

Define Population:

We use the potentially observable population concept of Rubin in formulating the problem. We assume that each unit could be potentially observed under each of the treatments, and represent the response for unit \( s \) given treatment \( a \) by the non-stochastic value \( y_{sa} \). The mean for treatment \( a \) is defined by \( \mu_a = \frac{1}{N} \sum_{s=1}^{N} y_{sa} \). The mean for subject \( s \) is defined by \( \mu_s = \frac{1}{A} \sum_{a=1}^{A} y_{sa} \). The overall mean over all the potential observable values is \( \mu = \frac{1}{NA} \sum_{a=1}^{A} \sum_{s=1}^{N} y_{sa} \).

The model we define is \( y_{sa} = \mu + \beta_s + \alpha_a + \varepsilon_{sa} \). First we consider \( \varepsilon_{sa} = 0 \), so we simplify the model as \( y_{sa} = \mu + \beta_s + \alpha_a \) where \( \beta_s = \mu_s - \mu \) refer to effect of units, \( \alpha_a = \mu_a - \mu \) refer to effect of treatments.

Parameterization:

Representing the \( N \times 1 \) potentially observable responses for treatment \( a \) by \( y_a \), we represent the \( N \times A \) matrix of potentially observable responses as \( y = (y_1, y_2, \ldots, y_A) \). The vector of treatment parameters is given by \( \mu = \left( \frac{1}{N} \mathbf{1}'_N y \right)' \). We define the variance of the potentially observable response as \( \frac{N-1}{N} \Sigma \) where \( \Sigma = (\sigma_{aa}) = \frac{1}{N-1} y' P_N y \) where \( P_N = \mathbf{I}_N - \frac{1}{N} J_N \).

An experiment will result in observing response for the units assigned to each treatment. We represent the experiment by a set of random variables that represent the units that could be potentially observed. We’ve already found the predictor correspond to a permutation of units. Further analysis now is conducted with respect to developing the predictor adding a permutation of treatments. In developing a solution, we highlight the basic notation first.

Explicitly, such a permutation can be defined in terms of indicator random variables \( U_i \). Let \( i = 1, \ldots, N \) index the position of a subject in a permutation, \( s = 1, \ldots, N \) be the label of a
subject. So \( U_{is} \) takes on a value of one if the subject \( s \) is assigned to position \( i \) in a permutation, or zero otherwise. Defining a matrix of such indicator random variables as \( U = \begin{pmatrix} U_{is} \end{pmatrix} \), \( Y = \begin{pmatrix} (Y_{ia}) \end{pmatrix} = UY \) represents a random permutation of units for the potentially observable population. To mimic random assignment of \( n \) units to each treatment, we assume that all permutations are equally likely (implying that \( E_{\xi} (U_{is}) = \frac{1}{N} \), and \( E_{\xi} (U_{is}U_{i's}) = \frac{1}{N} \) when \( i = i^* \) and \( s = s^* \), \( E_{\xi} (U_{is}U_{j's}) = \frac{1}{N(N-1)} \) when \( i \neq i^* \) and \( s \neq s^* \), and zero otherwise). Based on this definition we can calculate \( \text{Var}(U_{is}) = \frac{1}{N} - \frac{1}{N^2} \), \( \text{Cov}(U_{is},U_{i's}) = \frac{1}{N} - \frac{1}{N^2} \) when \( i = i^* \) and \( s = s^* \), \( \text{Cov}(U_{is},U_{i's}) = \frac{1}{N(N-1)} - \frac{1}{N^2} \) when \( i \neq i^* \) and \( s \neq s^* \), \( \text{Cov}(U_{is},U_{i's}) = -\frac{1}{N^2} \) otherwise. We refer to a randomly permuted unit as a primary sampling unit (PSU), and index the PSUs in \( Y \) by \( i = 1,\ldots,N \). Thus, the random variable representing response for PSU \( i \) given treatment \( a \) is defined by \( Y_{ia} = \sum_{s=1}^{N} U_{is}Y_{ia} \). Without loss of generality, we assume that an experiment consists of observing response for PSUs \( i = 1,\ldots,n \) with treatment \( a = 1 \), PSUs \( i = n+1,\ldots,2n \) with treatment \( a = 2 \), etc. The last treatment corresponds to response for PSUs \( i = (A-1)n + 1,\ldots,An \) with treatment \( a = A \).

Considering a permutation of treatments, we define indicator random variables \( V_{ja} \), \( j = 1,\ldots,A \), index the position of a treatment in a treatment permutation, \( a = 1,\ldots,A \) labels of the treatments. So \( V_{ja} \) takes on a value of one if the treatment \( a \) is assigned to position \( j \) in a permutation, or zero otherwise. Similarly, we defined a matrix of indicator random variables as \( V = \begin{pmatrix} (V_{ja}) \end{pmatrix} \). Assume that all permutations are equally likely (implying that \( E_{\xi} (V_{ja}) = \frac{1}{A} \), and \( E_{\xi} (V_{ja}V_{ja^*}) = \frac{1}{A} \) when \( j = j^* \) and \( a = a^* \), \( E_{\xi} (V_{ja}V_{ja^*}) = \frac{1}{A(A-1)} \) when \( j \neq j^* \) and \( a \neq a^* \), and zero otherwise). Then we can derive \( \text{Var}(V_{ja}) = \frac{1}{A} - \frac{1}{A^2} \), \( \text{Cov}(V_{ja},V_{ja^*}) = \frac{1}{A} - \frac{1}{A^2} \) when \( j = j^* \) and \( a = a^* \), \( \text{Cov}(V_{ja},V_{ja^*}) = \frac{1}{A} \frac{1}{A-1} - \frac{1}{A^2} \) when \( j \neq j^* \) and \( a \neq a^* \) and, \( \text{Cov}(V_{ja},V_{ja^*}) = -\frac{1}{A^2} \) otherwise.

We assume the independence of permutation of treatments and permutation of the units. So \( E(V_{ja}U_{is}) = \frac{1}{NA} \), for all \( j,a = 1,\ldots,A \), and
\[ i, s = 1, \ldots, N, \quad \text{Cov}(V_{ja}U_{is}, V_{ja}U_{is}) = \frac{1}{NA} \quad \text{when} \quad i = i^* \quad \text{and} \quad s = s^*, \quad j = j^* \quad \text{and} \quad a = a^*, \]

\[ \text{Cov}(V_{ja}U_{is}, V_{ja}U_{is}) = \frac{1}{AN(A-1)(N-1)} - \frac{1}{(NA)^2} \quad \text{when} \quad i \neq i^* \quad \text{and} \quad s \neq s^*, \quad j \neq j^* \quad \text{and} \quad a \neq a^*, \]

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\[ \text{Cov}(V_{ja}U_{is}, V_{ja}U_{is}) = -\frac{1}{(NA)^2} \quad \text{otherwise.} \]

A Random Permutation for the Potentially Observable Population Combined with a Random Permutation for the Treatments.

We express a random permutation model for the population as \( Y = (Y_{is}) = U_y \). When adding permutations of treatments, we develop a matrix as \( Y^* = (Y^*_{ij}) = U_y V' \), where \( Y^*_{ij} \) indicates the response of potential observable units in the \( i \)-th position of unit permutation and the \( j \)-th position of treatment permutation.

Take the column expansion:

\[ \text{vec}(Y') = \text{vec}(U_y V') = (Vy' \otimes I_N) \text{vec}(U), \]

Obviously, \( E(\text{vec}(Y')) = \sum_{sa}^{X} \sum_{a}^{A} \mu'_{Na} I_{Na} = \mu'_{Na} \]

\[ \text{Var}_{Uy} \left( \text{vec}(Y') \right) = \text{Var}_{Uy} \left( \text{vec}(Y') \right) + \text{Var}_{Uy} \left( \text{vec}(Y') \right) + \text{Var}_{Uy} \left( \text{vec}(Y') \right) \]

First, try to find the part:

\[ \text{Var}_{Uy} \left[ E_{Uy} \left( \text{vec}(Y') \right) \left[ E_{Uy} \left( \text{vec}(Y') \right) \right] \right. \]

\[ E_{Uy} \left( \text{vec}(Y') \right) = E_{Uy} \left( (Vy' \otimes I_N) \text{vec}(U) \right) = (Vy' \otimes I_N) E_{Uy} \left( \text{vec}(U) \right) \]

\[ = (Vy' \otimes I_N) \frac{1}{N} J_{NN} y = (Vy' \otimes I_N) \frac{1}{N} \text{vec}(J_N) = \text{vec} \left( I_N \frac{1}{N} J_N y V' \right) \]

\[ \text{Var}_{Uy} \left[ E_{Uy} \left( \text{vec}(Y') \right) \right] = \text{Var}_{Uy} \left[ \left( \frac{1}{N} I_N \otimes J_N y \right) \text{vec}(V') \right] \]

\[ = \left( \frac{1}{N} I_N \otimes J_N y \right) \text{Var}_{Uy} \left( \text{vec}(V') \right) \left( \frac{1}{N} I_N \otimes J_N y \right)^\prime \]
Similar as \( \text{Var} \left( \text{vec}(U) \right) \), we can derive that
\[
\text{Var} \left( \text{vec}(V) \right) = \frac{1}{A-1} \left( I_A - \frac{J_A}{A} \right) \otimes \left( I_A - \frac{J_A}{A} \right)
\]

We define \( P_A = I_A - \frac{J_A}{A} \), so we can simplify \( \text{Var} \left( \text{vec}(V) \right) = \frac{1}{A-1} \left( P_A \otimes P_A \right) \)

So \( \text{Var} \left( \text{vec}(V') \right) = \text{Var} \left( \text{vec}(V) \right) = \frac{1}{A-1} \left( P_A \otimes P_A \right) \)

\[
\text{Var}_y \left[ E_{y'P} \left( \text{vec}(Y') \right) \right]
\]
\[
= \left( \frac{1}{N} I_A \otimes J_N y \right) \frac{1}{A-1} \left( P_A \otimes P_A \right) \left( \frac{1}{N} I_A \otimes J_N y \right)'
\]
\[
= \left( \frac{1}{N} I_A \otimes J_N y \right) \frac{1}{A-1} \left( P_A \otimes P_A \right) \left( \frac{1}{N} I_A \otimes y'J_N \right)
\]
\[
= \frac{1}{N^2} \frac{1}{A-1} \left( I_A P_A I_A \otimes J_N y P_A y'J_N \right) = \frac{1}{N^2} \frac{1}{A-1} \left( P_A \otimes J_N y P_A y'J_N \right)
\]

We define the variance of the potentially observable response as \( \frac{A-1}{A} \Sigma_a \) where

\[
\Sigma_a = \left( \left( \sigma_{aa*} \right) \right) = \frac{1}{A-1} y P_y y'
\]

So the formula above can be simplified as
\[
\text{Var}_y \left[ E_{y'P} \left( \text{vec}(Y') \right) \right]
\]
\[
= \frac{1}{N^2} \left( P_A \otimes J_N \Sigma_a J_N \right)
\]

For the second part:
\[
\text{Var}_{y'P} \left( \left( V' \otimes I_N \right) \text{vec}(U) \right) = \left( V' \otimes I_N \right) \text{Var}_{y'P} \left( \text{vec}(U) \right) \left( yV' \otimes I_N \right)
\]
\[
= \left( V' \otimes I_N \right) \frac{1}{N-1} \left( P_N \otimes P_N \right) \left( yV' \otimes I_N \right)
\]
\[
= \frac{1}{N-1} \left( V y P_N y \right) \left( yV' \otimes P_N \right)
\]
\[
= \left( V \Sigma V' \right) \otimes P_N
\]
\[
E_y \left( \text{Var}_{y'P} \left( \left( V' \otimes I_N \right) \text{vec}(U) \right) \right)
\]
\[
= E_y \left( \left( V \Sigma V' \right) \otimes P_N \right) = \left( E_y \left( V \Sigma V' \right) \right) \otimes P_N
\]
\[
= \left( \text{tr} \left( \Sigma \right) \right) \frac{1}{A-1} \left( P_A \otimes \frac{1}{A} \Sigma J_A - J_A \right) \otimes P_N
\]
\[
= \left( \text{tr} \left( \Sigma \right) - \frac{1}{A} \Sigma J_A \right) \frac{1}{A-1} P_A + \frac{1}{A} \Sigma J_A \otimes P_N
\]
\[ Var_{U|Y} \left( \text{vec}(Y^*) \right) = E_Y \left[ Var_{U|Y} \left( \text{vec}(Y^*) \right) \right] + Var_Y \left( \text{vec}(Y^*) \right) \]

\[ = \left( \frac{tr(\Sigma)}{A - 1} - \frac{1'\Sigma 1}{A} \right) \frac{1}{A - 1} P_A + \frac{1'\Sigma 1}{A} \frac{1}{A - 1} J_A A P_N \right) \otimes P_N + \frac{1}{N^2} \left( P_A \otimes J_N \Sigma J_N \right) \]

\[ = \left( \frac{tr(\Sigma)}{A - 1} - \frac{1'\Sigma 1}{A} \right) \frac{1}{A - 1} P_A \otimes P_N + \frac{1'\Sigma 1}{A} \frac{1}{A - 1} J_A \otimes P_N + \frac{1}{N^2} \left( \frac{1'\Sigma J_N}{N^2} \right) P_A \otimes J_N \]