Inference for a One-Way Factorial Experiment  
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Introduction  
In this work we consider the one-way factorial experiment in a finite population setup where we have \( T \) levels (that will be referred as treatments) and \( M \) subjects (experimental units). A total of \( tm \leq M \) units will be randomly selected: \( m \) different subjects for each one of the \( t \) selected treatments. Both subjects and treatments will be selected via a simple random without replacement procedure.  

Population and Sampling  
The finite population consists of all the potentially observable responses for each subject under each treatment. The \( TM \) potentially observable responses are assumed to be fixed real numbers, measured without error, and are represented by  
\[
y' = (y'_1, \ldots, y'_T) \quad \text{with} \quad y'_t = (y_{1h}, \ldots, y_{Mh}) 
\]
where \( y_{th}, t = 1, \ldots, T \) and \( h = 1, \ldots, M \), is the response for subject \( h \) when given treatment \( t \), \( y_t \) is the \((M \times 1)\) vector of the responses for all the \( M \) subjects when given treatment \( t \) and \( y \) is the \((TM \times 1)\) vector of all the responses.  
The population parameters of interest would be the mean treatment effects  
\[
\mu_t = \frac{\sum_{h=1}^{M} y_{th}}{M}
\]
and the overall mean effect  
\[
\mu = \frac{\sum_{t=1}^{T} \sum_{h=1}^{M} y_{th}}{TM} = \frac{\sum_{t=1}^{T} \mu_t}{T}
\]
The variance and covariance parameters are given by  
\[
\Sigma = \begin{pmatrix} \sigma^2_1 & \cdots & \sigma^2_T \\ \vdots & \ddots & \vdots \\ \sigma^2_1 & \cdots & \sigma^2_T \end{pmatrix}
\]
the within-treatment covariance matrix, where  
\[
\sigma^2_t = \frac{\sum_{h=1}^{M} (y_{th} - \mu_t)^2}{M-1} (t = 1, \ldots, T)
\]
is the within-treatment variance, which are assumed to be the same for all the T treatments (homogeneity), and

\[ \sigma_{bs} = \frac{\sum_{t=1}^{M} (y_{isb} - \mu_{t})(y_{isb} - \mu_{t})}{M-1} \quad (t, s = 1, \ldots, T, t \neq s) \]

is the within-treatment covariances, which are assumed to be the same for all pairs of treatments.

The between-treatment variance is given by

\[ \sigma^2 = \frac{\sum_{t=1}^{T} (\mu_{t} - \mu)^2}{T-1} \]

**Sampling and Modeling**

t out of the T treatments will be randomly selected and for each one, m out of the M subjects will also be randomly selected. Each subject will be assigned to only one treatment. In both cases, the selections will be without replacement. After assigning an subject to a treatment, the corresponding value is observed without error.

As a first step in modeling the sampling procedure, we define a rearranged population, where the order of the T treatments and the M units have been randomly permutated, generating the random variable vector

\[ Y' = (Y'_1 \ldots Y'_T) \quad \text{with} \quad Y'_j = (Y_{j1}, \ldots, Y_{jM})' \]

where \( Y_{jg}, j = 1, \ldots, T \) and \( g = 1, \ldots, M \), is the response for the subject selected to position \( g \) when given treatment selected to position \( j \), \( Y_j \) is the (Mx1) random vector of the responses for all the M randomly assigned subjects when given randomly assigned treatment \( j \) and \( Y \) is the (TMx1) random vector of all the responses, which can be writing as

\[ Y = (U^T \otimes U^M)y = (U^T \otimes I_M)(U \otimes U^M)y \]

where \( U^T \) is a (TxT) random permutation matrix for the treatments given by

\[
U^T = \begin{pmatrix}
U_{11}^T & U_{12}^T & \cdots & U_{1T}^T \\
\vdots & \vdots & \ddots & \vdots \\
U_{j1}^T & U_{j2}^T & \cdots & U_{jT}^T \\
\vdots & \vdots & \ddots & \vdots \\
U_{T1}^T & U_{T2}^T & \cdots & U_{TT}^T 
\end{pmatrix}
\]
$U^M$ is a $(M \times M)$ random permutation matrix for the subjects given by

$$U^M = \begin{pmatrix}
U^M_{11} & U^M_{12} & \cdots & U^M_{1M} \\
U^M_{21} & U^M_{22} & \cdots & U^M_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
U^M_{M1} & U^M_{M2} & \cdots & U^M_{MM}
\end{pmatrix},$$

and $I_a$ is an identity matrix of order $a$. The random permutation matrices $U^T$ and $U^M$ are independently distributed, in the sense that treatment and subject are independently selected from each other. In this setup,

$$Y_{jj} = \sum_{j=1}^{M} \sum_{s=1}^{T} U^T_{js} U^M_{sj} y_{ss}$$

The last step in modeling the sampling procedure is to define the observed $(t \times m)$ sample vector $X = (X_1' \cdots X_t')'$, with $X_1' = (Y_{1,1} \cdots Y_{1,m})$, $X_2' = (Y_{2,1} \cdots Y_{2,2m})$, etc., composed of the first $m$ elements of $Y_1$, the second $m$ elements of $Y_2$, etc. Each $X_j$ can be defined as the product of a selection matrix and the corresponding $Y_j$, in the following way:

$$\forall j = 1, \ldots, t : \quad X_j = \begin{pmatrix} 0 & I_m & 0 \\ \end{pmatrix}_{M \times m} Y_j$$

Example: $T=3$, $M=7$, $t=2$ and $m=3$

$Y' = (Y_1', \ldots, Y_3')$ with $Y_j' = (Y_{j1}, Y_{j2}, Y_{j3})$

$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \end{pmatrix} \times Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \end{pmatrix}$

$X_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \end{pmatrix} \times Y_2 = \begin{pmatrix} Y_{24} \\ Y_{25} \\ Y_{26} \end{pmatrix}$

and the sample vector will be

$X' = (Y_{11}, Y_{12}, Y_{13}, Y_{24}, Y_{25}, Y_{26})$
Preliminary Results on First and Second Moments

For the treatment permutation matrix $U^T$

$$E(U^T_{jj'}) = \frac{1}{T} (j, j' = 1, \ldots, T)$$

$$E(U^T_{jj'}U^T_{kk'}) = \begin{cases} 
\frac{1}{T(T-1)} & j \neq k \text{ and } j' \neq k' \\
\frac{1}{T} & j = k \text{ and } j' = k' \\
0 & \text{otherwise}
\end{cases}$$

$$E(U^T) = \frac{J^T_T}{T}, \text{ where } J^T_T \text{ is a (T} \times T\text{) matrix of one's.}$$

For the subject permutation matrix $U^M$

$$E(U^M_{gg'}) = \frac{1}{M} (g, g' = 1, \ldots, M)$$

$$E(U^M_{gg'}U^M_{ll'}) = \begin{cases} 
\frac{1}{M(M-1)} & g \neq l \text{ and } g' \neq l' \\
\frac{1}{M} & g = l \text{ and } g' = l' \\
0 & \text{otherwise}
\end{cases}$$

$$E(U^M) = \frac{J^M_M}{M}, \text{ where } J^M_M \text{ is a (M} \times M\text{) matrix of one's.}$$

The Population Mean

$$E(Y) = E(U^T \otimes U^M)Y = (E(U^T) \otimes E(U^M))Y$$

$$= (\frac{J^T_T}{T} \otimes \frac{J^M_M}{M})Y = \frac{1}{TM} J^T_M Y = 1^T_M \mu$$
where \( 1_a \) is an \((a \times 1)\) column vector of ones. Each one of the TM elements of \( Y \) has expected value \( \mu \).

**The Population Variance**

\[
Y_j' Y_k' = \left[ \sum_{t=1}^{T} \sum_{s=1}^{M} U_{js}^T U_{ts}^M y_{ts} \right] \quad : \quad \left[ \sum_{t=1}^{T} \sum_{s=1}^{M} U_{js}^T U_{ts}^M y_{ts} \right] \quad \ldots \quad \left[ \sum_{t=1}^{T} \sum_{s=1}^{M} U_{js}^T U_{Ms}^M y_{ts} \right]
\]

1. The expected value of the \( i \)-th element, \( i=1, \ldots, M \), in the main diagonal

\[
E \left\{ \left( \sum_{t=1}^{T} \sum_{s=1}^{M} U_{js}^T U_{ts}^M y_{ts} \right) \left( \sum_{t=1}^{T} \sum_{s=1}^{M} U_{js}^T U_{ts}^M y_{ts} \right) \right\} = \sum_{t=1}^{T} \sum_{s=1}^{M} \sum_{t'=1}^{T} \sum_{s'=1}^{M} E(U_{jts}^T U_{kts'}^T) E(U_{lit}^M U_{lit}^M) y_{tis} y_{tis'}
\]

\[
= \sum_{t=1}^{T} \sum_{t'=1}^{T} \sum_{s=1}^{M} E(U_{jts}^T U_{kts'}^T) E(U_{lit}^M U_{lit}^M) y_{tis} y_{tis'}
\]

(noting that \( U_{lit}^M U_{lit}^M = 0 \) for \( s \neq s' \))

\[
= \frac{1}{M} \sum_{t=1}^{T} \sum_{t'=1}^{T} \sum_{s=1}^{M} E(U_{jts}^T U_{kts'}^T) y_{tis} y_{tis'}
\]